

Math 1A Fall 2001: Section 4.5 Solutions

2. (a) $\lim_{x \rightarrow a} [f(x)p(x)]$ is an indeterminate form of type $0 \cdot \infty$.

(b) When x is near a , $p(x)$ is large and $h(x)$ is near 1, so $h(x)p(x)$ is large. Thus, $\lim_{x \rightarrow a} [h(x)p(x)] = \infty$.

(c) When x is near a , $p(x)$ and $q(x)$ are both large, so $p(x)q(x)$ is large. Thus, $\lim_{x \rightarrow a} [p(x)q(x)] = \infty$.

$$6. \lim_{x \rightarrow 1} \frac{x^a - 1}{x^b - 1} \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$$

$$14. \lim_{x \rightarrow \infty} \frac{e^x}{x^3} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{3x^2} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{6x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{6} = \infty$$

$$22. \lim_{x \rightarrow -\infty} x^2 e^x = \lim_{x \rightarrow -\infty} \frac{x^2}{e^{-x}} \stackrel{H}{=} \lim_{x \rightarrow -\infty} \frac{2x}{-e^{-x}} \stackrel{H}{=} \lim_{x \rightarrow -\infty} \frac{2}{e^{-x}} = \lim_{x \rightarrow -\infty} 2e^x = 0$$

$$28. \lim_{x \rightarrow 0} (\csc x - \cot x) = \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) = \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\sin x}{\cos x} = 0$$

32. $y = (\sin x)^{\tan x} \Rightarrow \ln y = \tan x \ln(\sin x)$, so

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \tan x \ln(\sin x) = \lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{\cot x} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{(\cos x)/\sin x}{-\csc^2 x} = \lim_{x \rightarrow 0^+} (-\sin x \cos x) = 0 \Rightarrow$$

$$\lim_{x \rightarrow 0^+} (\sin x)^{\tan x} = \lim_{x \rightarrow 0^+} e^{\ln y} = e^0 = 1.$$

56. Let the radius of the circle be r . We see that $A(\theta)$ is the area of the whole figure (a sector of the circle with radius 1), minus the area of $\triangle OPR$. But the area of the sector of the circle is $\frac{1}{2}r^2\theta$ (see endpapers), and the area of the triangle is $\frac{1}{2}r|PQ| = \frac{1}{2}r(r \sin \theta) = \frac{1}{2}r^2 \sin \theta$. So we have
 $A(\theta) = \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta = \frac{1}{2}r^2(\theta - \sin \theta)$. Now by elementary trigonometry,
 $B(\theta) = \frac{1}{2}|QR||PQ| = \frac{1}{2}(r - |OQ|)|PQ| = \frac{1}{2}(r - r \cos \theta)(r \sin \theta) = \frac{1}{2}r^2(1 - \cos \theta) \sin \theta$. So the limit we want is

$$\begin{aligned} \lim_{\theta \rightarrow 0^+} \frac{A(\theta)}{B(\theta)} &= \lim_{\theta \rightarrow 0^+} \frac{\frac{1}{2}r^2(\theta - \sin \theta)}{\frac{1}{2}r^2(1 - \cos \theta) \sin \theta} \stackrel{H}{=} \lim_{\theta \rightarrow 0^+} \frac{1 - \cos \theta}{(1 - \cos \theta) \cos \theta + \sin \theta (\sin \theta)} \\ &= \lim_{\theta \rightarrow 0^+} \frac{1 - \cos \theta}{\cos \theta - \cos^2 \theta + \sin^2 \theta} \stackrel{H}{=} \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{-\sin \theta - 2 \cos \theta (-\sin \theta) + 2 \sin \theta (\cos \theta)} \\ &= \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{-\sin \theta + 4 \sin \theta \cos \theta} = \lim_{\theta \rightarrow 0^+} \frac{1}{-1 + 4 \cos \theta} = \frac{1}{-1 + 4 \cos 0} = \frac{1}{3} \end{aligned}$$