Exam Two Review — Integration - Part 2 — Fall '99 Math 1B

Topics left to cover:

Integration Applications

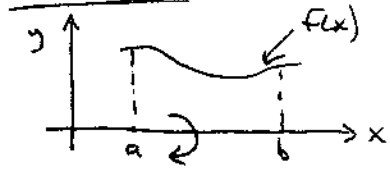
5. Volume Integrals
6. Arc Length
7. Surface Area
8. Work Integrals
9. Fluid Force

5 Volume Integrals

all volume integrals can be solved by slicing up, calculating $\underbrace{\text{cross-section areas}}$ and integrating:

$$\int_a^b CSA\, dx$$

For instance: Volume of Rotation



Volume of region created by rotating curve $y=f(x)$ around x-axis between $x=a$ and $x=b$ is $\int_a^b \pi (f(x))^2\, dx$

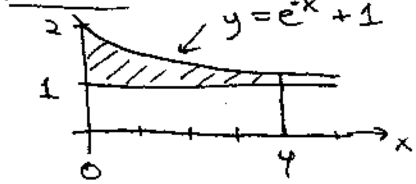
Example 1 calculate volume of object given by rotating $y=e^x$ around x-axis between $x=0$ and $x=5$.

Remember you might end up with a "harder" integral, i.e. you have to use int. by parts, or partial fractions – that's fine! just do it!

Example 2 calculate volume of object given by rotation of $y = \sqrt{\ln x}$ around x-axis, $1 \leq x \leq 10$

Twist on same theme: cross-sections might be washers, that's fine, just subtract out inside circle: instead of just $\int \pi (f(x))^2 dx$ you have $\int \Big(\underbrace{\pi (f_{out}(x))^2}_{\text{outer circle}} - \underbrace{\pi (f_{in}(x))^2}_{\text{inside circle}} \Big) dx$

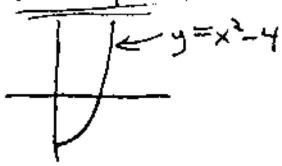
Example 3



Calculate the volume given by rotating the area shown around the x-axis

Another variant: Rotating around y-axis. Just solve $y = f(x)$ for x, get $x = g(y)$, then do everything in terms of y

Example 4 Calculate volume of object created by rotating $y = x^2 - 4$ around y-axis for $0 \leq y \leq 4$

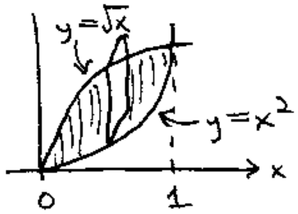


⑤ Volume cont

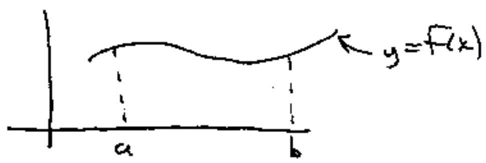
Pop-ups!

Same idea — need to calculate Cross-Section area (which won't necessarily be a circle, it might be a square, or a triangle)

Example 5 Calculate volume of object which has base as shown, and whose cross-sections perpendicular to the x-axis are squares.



⑥ Arc-length: Know the formula!!

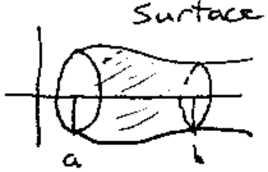


Note - only a few specially rigged functions end up giving you integrals you can solve, so look for them — most often $1+(f'(x))^2$ becomes a perfect square

Example 6 Calculate arc-length along the curve $y=\frac{x^7}{7}+\frac{x^{-5}}{20}$ between $x=1$ and $x=3$

⑦ Surface Area: Know the formula(s)!!

If $y=f(x)$ is rotated around the x-axis then the surface area of the resulting object between $x=a$ and $x=b$ is...



(7) Surface Area cont

<u>Example 7.</u> Find the surface area of the object formed by rotating $y = 3x^{1/2}$ around the x-axis for $0 \le x \le 2$
(or "on the interval $[0,2]$")

Around the y-axis just switch x for y in formula: $\int_c^d 2\pi\, g(y)\sqrt{1+(g'(y))^2}\,dy$

<u>Example 8</u> Find the surface area of the object formed by rotating $x = \frac{1}{3}y^3 + \frac{1}{4}y^{-1}$ around the y-axis for $1 \le y \le 2$

(8) Work!!

for a constant force F (like gravitational pull on an object, i.e. its weight)

$$\text{Work} = F \cdot d$$

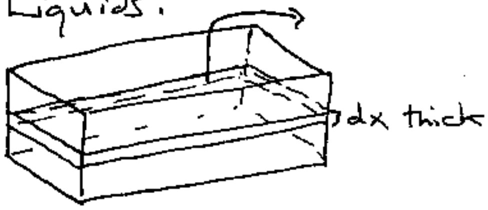
usual units: ft-lb $\qquad$ lb $\quad$ ft

for a changing force $F(x)$, $\quad W = \int_a^b F(x)\,dx$

for example a Spring – variable force $F(x) = \underbrace{k}_{\text{spring constant}} x$

<u>Example 9</u> calculate the work done in pulling a spring from its natural resting spot to a 2 foot extension. Assume $k = 10$.

Pumping Liquids!

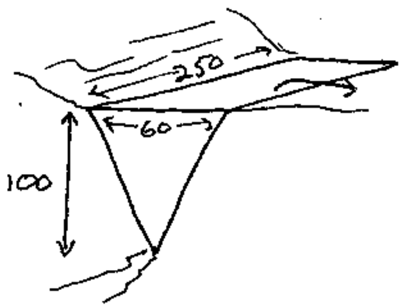


Total work pumping out tank is found by summing up work taken to lift or pump up one thin slab of water (or liquid) up to whatever height is given.

Approach:

1. Calculate cross-section area (in picture, for example - a rectangle) ↗ might vary at different depths)
2. Write down depth of the slab
3. Evaluate the integral $\int (\text{depth})\, \rho \cdot CSA\, dx$

Example 10.



Calculate total work done to pump the dam in the picture dry. I.e. pump all the water to the level of the top of the dam. The dam is 100 ft. high, 60 ft. across, and the lake is 250 ft. long. Assume density is $\rho = 62.4$

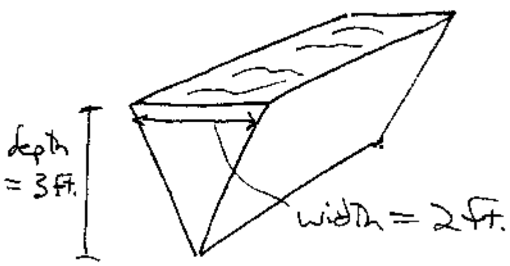
(9) Fluid Force!

Note the formulas for pumping liquids out of tanks, etc., are extremely similar to the ones used for calculating fluid force!

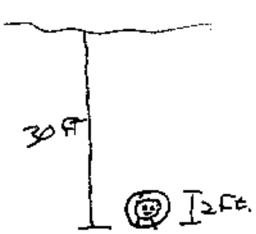
Pumping Liquids:

Fluid Force:

Example 11 A watering trough is filled with water. What is the fluid force on one triangular end of the trough? Assume ρ = density of water = 64 lbs/ft^3.



Example 12 As the Titanic slowly settles in to its watery grave Leo D. stares out of a porthole, 2 feet in diameter. If the bottom of the porthole is currently 30 feet under the water's surface, then set up an integral giving the fluid force on the porthole. Assume density is 64 lbs/ft^3



Answers

Example 1 $\int_0^5 \pi e^{2x} = \left(\frac{1}{2}\pi\, e^{2x}\right)\Big|_0^5 = \frac{\pi}{2} e^{10} - \frac{\pi}{2} e^0 = \frac{\pi}{2}\left(e^{10}-1\right)$

you probably can just see this directly,
if not, sub $u = 2x$, $du = 2dx$, $\frac{1}{2}du = dx$, etc.

Example 2 $\int_1^{10} \pi \left(\sqrt{\ln x}\right)^2 dx = \pi \int_1^{10} \ln x \; dx$

by parts trick, let $u = \ln x$ $\quad dv = dx$
$du = \frac{1}{x}dx \quad v = x$

then $\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x}\, dx$

$= x \ln x - x$

so $\pi \int_1^{10} \ln x \, dx = \pi \left(x \ln x - x \Big|_1^{10} \right)$

$= \pi \left(10 \ln 10 - 10 - (1 \ln 1 - 1)\right)$

$= \pi \; (10 \ln 10 - 9)$

Example 3 $\int_0^4 \left(\pi(e^{-x}+1)^2 - \pi(1)^2\right)dx = \int_0^4 \left(\pi(e^{-2x}+2e^{-x}+1) - \pi\right)dx = \int_0^4 \pi\left(e^{-2x}+2e^{-x}\right)dx$

$= \left(-\frac{\pi}{2}e^{-2x} - 2\pi e^{-x}\right)\Big|_0^4 = -\frac{\pi}{2}e^{-8} - 2\pi e^{-4} - \left(-\frac{\pi}{2} - 2\pi\right)$

$= \frac{5\pi}{2} - \pi\left(\frac{e^{-8}}{2} + 2e^{-4}\right)$

Example 4 $y = x^2 - 4$, so $y + 4 = x^2$, $\sqrt{y+4} = x$,

get $\int_0^4 \pi \left(\sqrt{y+4}\right)^2 dy = \int_0^4 \pi(y+4)\,dy = \left(\pi \frac{y^2}{2} + 4\pi y\right)\Big|_0^4$

$= \frac{\pi}{2}16 + 16\pi - 0 = 24\pi$

Answers continued

Example 5 squares have sides of length $\sqrt{x} - x^2$, area $= (\sqrt{x} - x^2)^2$

so get $\int_0^1 (\sqrt{x} - x^2)^2\,dx = \int_0^1 (x - 2\sqrt{x}\,x^2 + x^4)\,dx = \int_0^1 (x - 2x^{5/2} + x^4)\,dx$

$$= \frac{x^2}{2} - \frac{4}{7}x^{7/2} + \frac{x^5}{5}\Big|_0^1 = \frac{1}{2} - \frac{4}{7} + \frac{1}{5} = \frac{9}{70}$$

Example 6 $y = \frac{x^7}{7} + \frac{x^{-5}}{20}$, so $\frac{dy}{dx} = x^6 - \frac{1}{4}x^{-6}$

$$\left(\frac{dy}{dx}\right)^2 = \left(x^6 - \frac{1}{4}x^{-6}\right)^2 = x^{12} - \frac{2}{4} + \frac{1}{16}x^{-12}$$

integral is $\int_1^3 \sqrt{1 + x^{12} - \frac{1}{2} + \frac{1}{16}x^{-12}}\,dx = \int_1^3 \sqrt{x^{12} + \frac{1}{2} + \frac{1}{16}x^{-12}}\,dx$

$$= \int_1^3 \sqrt{\left(x^6 + \frac{1}{4}x^{-6}\right)^2}\,dx = \int^3 \left(x^6 + \frac{1}{4}x^{-6}\right)dx$$

$$= \frac{x^7}{7} - \frac{x^{-5}}{20}\Big|_1^3 = \frac{3^7}{7} - \frac{1}{20 \cdot 3^5} - \left(\frac{1}{7} - \frac{1}{20}\right)$$

Example 7 $y = f(x) = 3 \cdot x^{1/2}$

so $f'(x) = \frac{3}{2}x^{-1/2}$ $\qquad 1 + (f'(x))^2 = 1 + \frac{9}{4x} = \frac{4x}{4x} + \frac{9}{4x}$

slight trick $\qquad = \frac{4x+9}{4x}$

so $2\pi \int_0^2 f(x)\sqrt{1 + (f'(x))^2}\,dx$

$$= 2\pi \int_0^2 3x^{1/2} \cdot \sqrt{\frac{4x+9}{4x}}\,dx = 2\pi \int_0^2 3x^{1/2} \frac{\sqrt{4x+9}}{2x^{1/2}}\,dx$$

$= 3\pi \int_0^2 \sqrt{4x+9}\,dx$ sub $u = 4x+9$ $\quad du = 4\,dx$, $\frac{1}{4}du = dx$

$$= \frac{3\pi}{4}\int_{x=0}^{x=2} u^{1/2}\,du = \frac{3\pi}{4}\left(\frac{2}{3}u^{3/2}\right)\Big|_{x=0}^{x=2}$$

$$= \frac{\pi}{2}(4x+9)^{3/2}\Big|_0^2$$

$$= \frac{\pi}{2}\left(17^{3/2} - 9^{3/2}\right) = \frac{\pi}{2}\left(17^{3/2} - 27\right)$$

Example 8 so if $g(y) = \frac{1}{3}y^3 + \frac{1}{4}y^{-1}$ then $g'(y) = y^2 - \frac{1}{4}y^{-2}$

so $1 + (g'(y))^2 = 1 + \left(y^4 \underline{- \frac{1}{2}} + \frac{1}{16}y^{-4}\right) = y^4 \underline{+ \frac{1}{2}} + \frac{1}{16}y^{-4} = \left(y^2 + \frac{1}{4}y^{-2}\right)^2$

so $\int_1^2 2\pi\, g(y)\sqrt{1 + (g'(y))^2}\, dy = \int_1^2 2\pi \left(\frac{1}{3}y^3 + \frac{1}{4}y^{-1}\right)\sqrt{\left(y^2 + \frac{1}{4}y^{-2}\right)^2}\, dy$

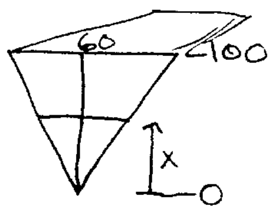
$= \int_1^2 2\pi \left(\frac{1}{3}y^5 + \left(\frac{1}{12} + \frac{1}{4}\right)y + \frac{1}{16}y^{-3}\right) dy = 2\pi \int_1^2 \left(\frac{1}{3}y^5 + \frac{1}{3}y + \frac{1}{16}y^{-3}\right) dy$

$= 2\pi \left[\frac{1}{18}y^6 + \frac{1}{6}y^2 - \frac{1}{32}y^{-2}\right]\Big|_1^2 = 2\pi\left(\left(\frac{64}{18} + \frac{4}{6} - \frac{1}{128}\right) - \left(\frac{1}{18} + \frac{1}{6} - \frac{1}{32}\right)\right)$

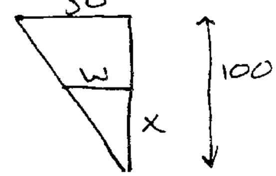
whatever this is!

Example 9 $\int_0^2 kx\, dx = \int_0^2 10x\, dx = \frac{10x^2}{2}\Big|_0^2 = 20$ ft. lbs of work

Example 10



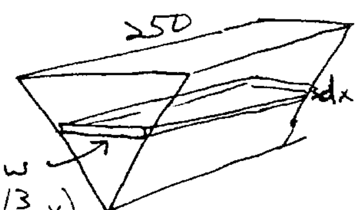
similar triangles



so $\frac{w}{x} = \frac{30}{100}$

$w = \frac{3}{10}x$

so now



$2w = 2\left(\frac{3}{10}x\right) = \frac{3}{5}x$

cross section is $\left(\frac{3}{5}x\right) \cdot 250 = 150x$

volume of slab is $150x\, dx$

depth of slab is $100 - x$

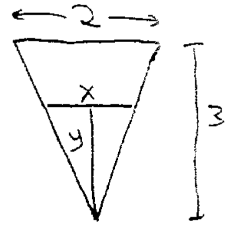
integral is simply $\int_0^{100} (100 - x) \cdot \rho\, 150x\, dx$

$= \rho \int_0^{100} (15{,}000x - 150x^2)\, dx = \rho \cdot \left(\frac{15{,}000x^2}{2} - \frac{150}{3}x^3\right)\Big|_0^{100}$

$=$ a big number.

Answers continued

Example 11



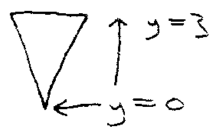
so similar triangles yields

$$\frac{x}{y} = \frac{2}{3}, \quad \text{so } x = \frac{2}{3}y$$

i.e. $\ell(y)$ = length across trough $= \frac{2}{3}y$

depth of water, $d(y) = 3-y$

(This is all assuming y measures from the tip of the trough up to the top, 3 feet up)

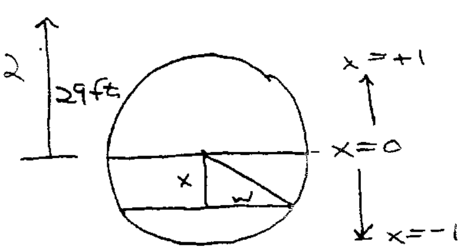


so total integral $= \int_0^3 \rho \cdot d(y) \cdot \ell(y)\, dy = \int_0^3 64 \cdot (3-y) \cdot \frac{2}{3} y\, dy$

$= 64 \int_0^3 \left(2y - \frac{2}{3}y^2\right) dy = 64\left(y^2 - \frac{2}{9}y^3\right)\Big|_0^3$

$= 64(9-6) = 64 \cdot 3 = 192$ lbs of force

Example 12



so $x^2 + w^2 = 1$

then $w = \sqrt{1-x^2}$

so total width across $\ell(x) = 2\sqrt{1-x^2}$

depth of water $d(x) = (29-x)$ check this by putting in $x=-1$, $x=1$ for instance

so fluid force $= \int_{-1}^{1} \rho\, (29-x)\sqrt{1-x^2}\, dx$

$= \int_{-1}^{1} 64\, (29-x)\sqrt{1-x^2}\, dx$