Harvard University
Math 1b. Second Semester Calculus
First Exam

Name Answer Key

October 27, 1999

First Exam

- Do not open this exam booklet until you are directed to do so. You will have two hours to complete this exam.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Do not put part of the answer to one problem on the back of the sheet for another problem.
- Don't spend too much time on any one problem. Read them all through first and work on them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat. Also, be sure to justify your solutions (unless you are explicitly told otherwise), so we can follow your reasoning.
- Concentrate and do well!

Problem	Points	Grade
1	9	9
2	8	8
3	12	12
4	14	14
5	12	12
6	15	15
7	6	6
8	9	9
9	12	12
bonus	3	3
Total	100	100

Please circle your section:

MWF 10	Curt McMullen	TTh 10	Yuhan Zha
MWF 10	Pete Clark	TTh 10	Yu-Ru Liu
MWF 11	Tammy Lefcourt	TTh 10	Nina Zipser
MWF 11	Greg Warrington	TTh 11:30	Andy Engelward
MWF 12	Kiril Selverov	TTh 11:30	Alexandru Popa

Problem 1: (9 points total) Determine whether each of the following series converges or diverges. If a series converges, find its sum.

a. **(3 points)** $\sum_{k=0}^{\infty} \left(\frac{2}{\pi}\right)^k$

This is a geometric series with ratio $r = \frac{2}{\pi}$. The series <u>converges</u> as $\left|\frac{2}{\pi}\right| < 1$,

$$\text{sum} = \frac{a}{1-r} \quad \text{with } a = 1^{st}\text{ term} = 1$$

$$= \frac{1}{1-\frac{2}{\pi}} \quad \text{or} \quad \frac{\pi}{\pi-2}$$

b. **(3 points)** $\sum_{k=2}^{\infty} \left(\frac{2}{\pi}\right)^k$

Again, this is a geometric series with ratio $\frac{2}{\pi}$, again <u>converges</u> as $|r| < 1$. Here the first term is $\left(\frac{2}{\pi}\right)^2$, so sum $= \frac{(4/\pi^2)}{\left(1-\frac{2}{\pi}\right)}$

$$= \frac{4}{\pi(\pi-2)}$$

c. **(3 points)** $\sum_{k=0}^{\infty} \left(\frac{\pi}{2}\right)^k$

Again, this is a geometric series, but now clearly $r = \frac{\pi}{2} > 1$, so the series <u>diverges</u>

Problem 2: (8 points total) For each of the following infinite series, determine if it converges or diverges. You need to justify your answer to receive full credit (you do not need to specify if convergence is conditional or absolute, just whether the series converges or diverges).

a. (4 points) $\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{k(k+10)}$

Note that the limit of the terms, $\lim_{k\to\infty} \left|\frac{(-1)^k k^2}{k(k+10)}\right| = \lim_{k\to\infty}\left(\frac{k^2}{k^2+10k}\right)$

$= 1 \neq 0$

so divergence test implies that the series $\boxed{\text{diverges}}$

b. (4 points) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+1)(k+2)}}$

Use limit comparison test with the p-series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$

Note $\lim_{k\to\infty}\left(\frac{1}{\sqrt{k(k+1)(k+2)}} \Big/ \frac{1}{k^{3/2}}\right)$

$= \lim_{k\to\infty}\left(\frac{\sqrt{k^3}}{\sqrt{k(k+1)(k+2)}}\right) = \lim_{k\to\infty}\left(\sqrt{\frac{k^3}{k(k+1)(k+2)}}\right)$

$= 1$

So since $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges (p-series $p = \frac{3}{2} > 1$)

then so does the series in question

$\boxed{\text{Converges}}$

Problem 3: (12 points total) For each of the following infinite series, determine if it converges or diverges. You need to justify your answer to receive full credit (you do not need to specify if convergence is conditional or absolute, just whether the series converges or diverges).

a. **(4 points)** $\sum_{k=1}^{\infty} \frac{(-1)^{(k+1)}}{\sqrt{4k}}$ $\boxed{\text{Converges}}$ by Alternating Series Test:

Note $|a_k| > |a_{k+1}|$: $\frac{1}{\sqrt{4k}} > \frac{1}{\sqrt{4(k+1)}}$ ✓

and $\lim_{k\to\infty} \left| \frac{(-1)^{k+1}}{\sqrt{4k}} \right| = 0$ ✓

b. **(4 points)** $\sum_{k=1}^{\infty} \frac{k^4+1}{k!}$ Can use the ratio test:

$$\lim_{k\to\infty} \left| \frac{(k+1)^4+1}{(k+1)!} \Big/ \frac{k^4+1}{k!} \right| = \lim_{k\to\infty} \left| \frac{((k+1)^4+1)\,k!}{(k+1)!\,(k^4+1)} \right|$$

$$= \lim_{k\to\infty} \left| \frac{(k+1)^4+1}{(k+1)(k^4+1)} \right|$$

$$= 0 < 1 \left(\begin{array}{l} \text{highest power of} \\ k \text{ in numerator} = 4 \\ \text{in denominator} = 5 \end{array} \right)$$

So series $\boxed{\text{converges}}$

c. **(4 points)** $\sum_{k=1}^{\infty} \frac{|\sin k|}{(k+3)^{\frac{3}{2}}}$

Note $|\sin k| \leq 1$, so can compare with

the p-series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ $\left(\text{i.e. } \frac{|\sin k|}{(k+3)^{3/2}} < \frac{1}{k^{3/2}} \right)$

and since $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges $\left(\text{p-series with } p = 3/2 > 1 \right)$

Problem 4: (14 points total) Find the interval of convergence and radius of convergence for each of the following power series (be sure to check convergence at endpoints).

a. (7 points) $\sum_{k=0}^{\infty} \frac{(x-5)^k}{\ln(k+2)}$

Use ratio test: $\lim_{k\to\infty} \left| \frac{(x-5)^{k+1}}{\ln((k+1)+2)} \cdot \frac{\ln(k+2)}{(x-5)^k} \right|$

$$= \lim_{k\to\infty} \left| (x-5) \cdot \frac{\ln(k+2)}{\ln(k+3)} \right| = |x-5|$$

So power-series converges if $|x-5| < 1$ or $-1 < x-5 < 1$

$4 < x < 6$

Check endpoints: if $x=4$ series is $\sum_{k=0}^{\infty} \frac{(-1)^k}{\ln(k+2)}$

alt. series test $\rightarrow \lim_{k\to\infty} \frac{1}{\ln(k+2)} = 0$

and $\frac{1}{\ln((k+1)+2)} < \frac{1}{\ln(k+2)}$ $\Rightarrow$ converges for $x=4$

if $x=6$ series is $\sum_{k=0}^{\infty} \frac{1}{\ln(k+2)}$

compare to $\sum_{k=1}^{\infty} \frac{1}{k}$, since $\frac{1}{\ln(k+2)} > \frac{1}{k}$

and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, then so does the series when $x=6$

So Int. of Con. is $4 \leq x < 6$

(Radius of Con. is 1)

b. (7 points) $\sum_{k=0}^{\infty} \frac{(-x)^k 2^k}{k!}$

Use ratio test:

$\lim_{k\to\infty} \left| \frac{(-x)^{k+1} 2^{k+1}}{(k+1)!} \cdot \frac{k!}{(-x)^k 2^k} \right|$

$= \lim_{k\to\infty} \left| \frac{x \cdot 2}{k+1} \right| = 0$

so converges for all x $\quad -\infty < x < \infty$

with radius of convergence $= \infty$

Problem 5: (12 points total)

a. (7 points total) Write out the first four non-zero terms of the Taylor series about $x = 0$ (*i.e.* the Maclaurin series) for the function $f(x) = \dfrac{x}{1+x}$.

Note $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ (Maclaurin series)

sub in $(-x)$ for x and get $\dfrac{1}{1-(-x))} = 1 + (-x) + (-x)^2 + (x)^3 + \cdots$

$= 1 - x + x^2 - x^3 + \cdots$

multiply by x : $\dfrac{x}{1+x} = x\left(1 - x + x^2 - x^3 + \cdots\right)$

$= x - x^2 + x^3 - x^4 + \cdots$

$\Rightarrow$ Maclaurin series for $\dfrac{x}{1+x}$

b. (5 points) Write out the first four non-zero terms of the Taylor series about $x = 0$ for $g(x) = \int f(x)dx$, assuming that $g(0) = 2$.

Integrating the series $x - x^2 + x^3 - x^4 + \cdots$

term by term gives the Taylor series

$\left(\dfrac{x^2}{2} - \dfrac{x^3}{3} + \dfrac{x^4}{4} - \dfrac{x^5}{5} + \cdots\right) + C$, Note $g(0)$

Note $g(0) = \left(\dfrac{0}{2} - \dfrac{0}{3} + \cdots\right) + C = 2 \Rightarrow C = 2$,

so first four terms are $2 + \dfrac{x^2}{2} - \dfrac{x^3}{3} + \dfrac{x^4}{4} - \cdots$

Problem 6: (15 points total) Consider the function $f(x) = \cos(x^3)$

a. (6 points) Write out the first four non-zero terms of the Taylor series about $x = 0$ (*i.e.* the Maclaurin series) for $f(x)$.

Since the Maclaurin series for $\cos(x)$ is

$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, then by subbing

in x^3 for x, we find the Maclaurin

series for $\cos(x^3)$: $1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \frac{(x^3)^6}{6!} + \cdots$

$$= 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots$$

b. (3 points) Does this Taylor series converge for $x = 10$? Justify your answer.

Sure, since we know the interval of convergence for the Maclaurin series for $\cos(x)$ is $(-\infty, \infty)$, then substituting in x^3 for x won't affect this, and the interval of conv. for $\cos(x^3)$ must also be $(-\infty, \infty)$, and so it specifically converges at $x=10$.

c. (6 points) Using the identity $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$, write out the first four non-zero terms of the Taylor series about $x = 0$ for $\cos^2 x$.

Note $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$\text{So } \cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$$

$$= 1 - \frac{4x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \cdots$$

$$\text{Then } \frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1}{2} + \frac{1}{2}\left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots\right)$$

$$= 1 - x^2 + \frac{x^4}{3} - \frac{2}{45}x^6 + \cdots$$

Many people worked with the ~~incorrect~~ identity $\frac{1}{2} - \frac{1}{2}\cos x$, but were awarded full credit for their work as long as it was consistent (even though it was technically incorrect!!)

Problem 7: (6 points total) Consider the following infinite series

$$1 + e^{-2x} + e^{-4x} + e^{-6x} + e^{-8x} + \ldots$$

a. **(3 points)** For what values of x does this series converge?

Note this is just a geometric series with ratio $r = e^{-2x}$, so converges iff $|e^{-2x}| < 1$

since $e^{-2x} > 0$, then converges iff $0 < e^{-2x} < 1$,

or $\ln(e^{-2x}) < \ln 1 \Rightarrow -2x < 0$, so whenever $\boxed{x > 0}$

b. **(3 points)** For the values of x for which the series converges, find the sum as a function of x.

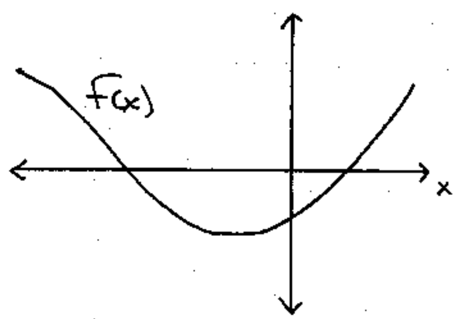
Then the sum is just $\frac{a}{1-r}$ with $a = 1$,

and $r = e^{-2x}$, so the sum is $\frac{1}{1-e^{-2x}}$

$\left(\text{or } \frac{e^{2x}}{e^{2x}-1} \right)$

Problem 8: (9 points total)

Let $f(x)$ be the function whose graph is given in the figure shown below.



recall how Taylor Series are created! around $x=0$:

$$\underset{a_0}{\underset{\parallel}{f(0)}} + \underset{a_1}{\underset{\parallel}{f'(0)}}\,x + \underset{a_2}{\underset{\parallel}{\frac{f''(0)}{2}}}\,x^2 + \ldots$$

Suppose the Taylor series about $x = 0$ for $f(x)$ is:

$$a_0 + a_1 x + a_2 x^2 + \ldots,$$

then circle the correct statement on each of the following three lines:

so since clearly $f(0) < 0$, $f'(0) > 0$, and concavity, $f''(0) > 0$, then

(1)	$a_0 < 0$	$a_0 = 0$	$a_0 > 0$
(2)	$a_1 < 0$	$a_1 = 0$	$a_1 > 0$
(3)	$a_2 < 0$	$a_2 = 0$	$a_2 > 0$

Problem 9: (12 points total)

a. (6 points) Write out the first four non-zero terms of the Taylor series about $x = \pi/2$ for $f(x) = \sin x$.

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

etc.

at $x = \frac{\pi}{2}$

$f(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = \underline{1}$

$f'(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) = 0$

$f''(\frac{\pi}{2}) = -\underline{1}$

$f'''(\frac{\pi}{2}) = 0$

etc.

so Taylor series starts as $f(\frac{\pi}{2}) + f'(\frac{\pi}{2})(x-\frac{\pi}{2}) + \frac{f''(\frac{\pi}{2})}{2!}(x-\frac{\pi}{2})^2 + \cdots$

$$1 + 0(x-\tfrac{\pi}{2}) + \frac{(-1)}{2!}(x-\tfrac{\pi}{2})^2 + \frac{0}{3!}(x-\tfrac{\pi}{2})^3 + \cdots$$

$$= 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} - \frac{(x-\frac{\pi}{2})^6}{6!} + \cdots$$

b. (6 points) Write out the first four non-zero terms of the Taylor series about $x = \pi/2$ for $\cos x$ by differentiating the above series.

Differentiating term by term yields the series.

$$0 - \frac{2}{2!}(x-\tfrac{\pi}{2})^1 + \frac{4}{4!}(x-\tfrac{\pi}{2})^3 - \frac{6}{6!}(x-\tfrac{\pi}{2})^5 + \frac{8}{8!}(x-\tfrac{\pi}{2})^7 + \cdots$$

$$= -(x-\tfrac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^3}{3!} - \frac{(x-\frac{\pi}{2})^5}{5!} + \frac{(x-\frac{\pi}{2})^7}{7!} - \cdots$$

(so this is the Taylor series about $x = \frac{\pi}{2}$ for $\cos(x)$ (and yes it does look similar to $\sin(x)$, because of the centering at $x = \frac{\pi}{2}$ for the series)