Tests For Convergence

Tests involving only one series, $A : \sum a_k$

Tests for which the terms may be positive, or negative, or both:

Test	What To Do	Result
Divergence Test	Let $L = \lim_{k\to\infty} a_k$.	If $L \neq 0$, A diverges.
Ratio Test for Absolute Convergence	Let $\rho = \lim_{k\to\infty} \frac{\lvert a_{k+1}\rvert}{\lvert a_k\rvert}$.	If $\rho < 1$, A converges absolutely. If $\rho > 1$, A doesn't converge absolutely. If $\rho = 1$, the test is inconclusive.
Root Test for Absolute Convergence	Let $\rho = \lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert}$.	If $\rho < 1$, A converges absolutely. If $\rho > 1$, A doesn't converge absolutely. If $\rho = 1$, the test is inconclusive.

Tests for which the terms must all be **positive**:

Ratio Test	Let $\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.	If $\rho < 1$, A converges. If $\rho > 1$, A diverges. If $\rho = 1$, the test is inconclusive.
Root Test	Let $\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$.	If $\rho < 1$, A converges. If $\rho > 1$, A diverges. If $\rho = 1$, the test is inconclusive.

Tests for which the terms must **alternate**:

Alternating Test	Check that $\lvert a_k\rvert > \lvert a_{k+1}\rvert$ for all k. Check that a_k and a_{k+1} always have opposite signs. Let $L = \lim_{k\to\infty} a_k$.	If $L = 0$, A converges.

Tests involving comparison of two series, $A : \sum a_k$ and $B : \sum b_k$, both of which have all **positive** terms

Comparison Test	Check that $a_k \leq b_k$ for all k.	If B converges then so does A. If A diverges then so does B.
Limit Comparison Test	Let $\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$.	If $0 < \rho < \infty$, then either both A and B converge, or both A and B diverge.

Some Special Series

Sums of powers of integers:

We have the following closed form expressions,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Geometric Series:

We can write any finite geometric series in closed form, as

$$\sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}.$$

If $|r| < 1$, then the infinite geometric series converges, and in fact

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

However, if $|r| \geq 1$, then $\sum_{k=0}^{\infty} ar^k$ diverges.

p–Series:

If $p > 1$, then $\sum\limits_{k=1}^{\infty} \frac{1}{k^p}$ converges.
However, if $p \leq 1$, then $\sum\limits_{k=1}^{\infty} \frac{1}{k^p}$ diverges.
As a particular example of divergence, notice that the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

diverges, since this is a p–series with $p = 1$. However, recall from the reverse side of this handout that the alternating test shows that the alternating harmonic series,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

is in fact convergent.