Math 1b HW Solutions (Week 10)

10.1

(2) Notice that $y' = x^3 - 2\sin x$ and that $y(0) = \frac{1}{4}(0)^4 + 2\cos(0) + 1 = 2 + 1 = 3$ ✓

(8) (a) Integ. Factors. $(p = -4x,\ q = 0)$

$$\mu = e^{\int -4x\,dx} = e^{-2x^2} \quad \frac{d}{dx}\left[e^{-2x^2} y\right] = 0 \text{ so } y = Ce^{2x^2}$$

$\left(\frac{d}{dx}(f(x,y)) = 0 \Rightarrow f(x,y) = \text{constant}\right)$

Separation of Variable

$$\frac{dy}{dx} = -4xy \text{ so } \int \frac{dy}{y} = \int -4x\,dx \Rightarrow \ln|y| = -2x^2 + c \Rightarrow |y| = e^{-2x^2+c}$$

$$\Rightarrow y = \pm e^c e^{-2x^2} \Rightarrow y = Ce^{-2x^2} \quad (C = \pm e^c)$$

(b) I F $(p = 1,\ q = 0)$

$$\mu = e^{\int dt} = e^t \quad \frac{d}{dt}\left[e^t y\right] = 0 \Rightarrow y = ce^{-t}$$

Sep. of var.

$$\int \frac{dy}{y} = \int -dt \Rightarrow \ln|y| = -t + k \Rightarrow |y| = e^{-t+k} \Rightarrow |y| = e^k e^{-t} \Rightarrow y = Ce^{-t} \quad (C = \pm e^k)$$

(10) $\frac{dy}{dx} = (1+y^2)x^2 \Rightarrow \int \frac{dy}{1+y^2} = \int x^2\,dx \Rightarrow \tan^{-1} y = \frac{x^3}{3} + C \Rightarrow y = \tan\left(\frac{x^3}{3} + C\right)$

(12) $(1+x^4)\frac{dy}{dx} = \frac{x^3}{y} \Rightarrow \int y\,dy = \int \frac{x^3}{1+x^4}\,dx \Rightarrow \frac{y^2}{2} = \frac{1}{4}\ln(1+x^4) + C \Rightarrow y = \pm\sqrt{\frac{1}{2}\left(\ln(1+x^4) + C\right)}$

(16) $y' - (1+x)(1+y^2) = 0 \Rightarrow \frac{dy}{1+y^2} = (1+x)\,dx \Rightarrow \tan^{-1} y = x + \frac{x^2}{2} + C$

$$\Rightarrow y = \tan\left(x + \frac{x^2}{2} + C\right)$$

(20) $\frac{dy}{dx} + 2xy = x \quad p = 2x \quad q = x$

$\mu = e^{\int 2x\,dx} = e^{x^2} \quad \frac{d}{dx}[ye^{x^2}] = xe^{x^2} \Rightarrow ye^{x^2} = \frac{1}{2}e^{x^2} + C$

$\Rightarrow y = \frac{1}{2} + Ce^{-x^2}$

(23) $(x^2+1)\frac{dy}{dx} + xy = 0 \Rightarrow p = \frac{x}{x^2+1} \quad q = 0$

$\mu = e^{\int \frac{x}{x^2+1}dx} = e^{\frac{1}{2}\ln(x^2+1)} = \sqrt{x^2+1} \quad \frac{d}{dx}[y\sqrt{x^2+1}] = 0$

$\Rightarrow y = \frac{C}{\sqrt{x^2+1}}$

(26) $\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = x\,dx \Rightarrow \ln|y| = x^2/2 + k \Rightarrow y = \pm e^k e^{x^2/2} \Rightarrow y = Ce^{x^2/2}$ in general

(a) $y(0) = Ce^{0/2} = 1 \Rightarrow C = 1$ so $y = e^{x^2/2}$

(b) $y(0) = Ce^{0/2} = 1/2 \Rightarrow C = 1/2$ so $y = \frac{1}{2}e^{x^2/2}$

(28) $\frac{dy}{dt} + y = 2 \quad p = 1 \quad q = 2$

$\mu = e^{\int dt} = e^t \quad \frac{d}{dt}[ye^t] = 2e^t \Rightarrow ye^t = 2e^t + k \Rightarrow y = 2 + \frac{k}{e^t}$

$y(0) = 1 = 2 + \frac{k}{e^0} \Rightarrow k = -1$ so $\underline{\underline{y = 2 - \frac{1}{e^t}}}$

(44) $\frac{dy}{dt} = 5 \cdot 10 - \frac{y}{200}(10)$ (y = amt. of salt in tank) $y(0) = 0$

so $\frac{dy}{dt} + \frac{1}{20}y = 50 \quad p = \frac{1}{20} \quad q = 50$

$\mu = e^{\int \frac{1}{20}dt} = e^{t/20} \quad \frac{d}{dt}[ye^{t/20}] = 50e^{t/20}$

$\Rightarrow ye^{t/20} = 1000e^{t/20} + C$

$\Rightarrow y = 1000 + C/e^{t/20}$

$y(0) = 0$ so $\frac{-C}{e^{0/20}} = 1000 \Rightarrow C = -1000$ so

(a) $y(t) = 1000 - \frac{1000}{e^{t/20}}$ lbs.

(b) $y(30) = 1000 - \frac{1000}{e^{3/2}} \approx 777$ lbs.

(45) The volume V, of the lake is $\pi(15)^2 \cdot 3)m^3 \cdot 264\,gal/m^3 = 166{,}050\pi$ gallons.

Let $y(t)$ = lbs of mercury salts at time t. Then $\frac{dy}{dt} = 0 - 10^3 \frac{y}{V} = \frac{-y}{166.05\pi}$

Since $y(0) = 10^{-5}V = 1.6605\pi$ lbs, $\frac{dy}{y} = \frac{-1}{166.05\pi}dt \Rightarrow \ln|y| = \frac{-t}{166.05\pi} + C$

$\Rightarrow y = c'e^{-t/166.05\pi}$ where $c' = \pm e^c$

$\Rightarrow y(0) = c'e^0 = 10^{-5}V$, $c' = 10^{-5}V \Rightarrow y = 1.6605\pi e^{-t/166.05\pi}$

t	1	2	3	4	5	6	7
$y(t)$	5.2066	5.1967	5.1867	5.1768	5.1669	5.1570	5.1471

t	8	9	10	11	12
$y(t)$	5.1372	5.1274	5.1176	5.1078	5.0980

p.9

(2) $y = A\cos t + B\sin t$ $y' = -A\sin t + B\cos t$ $y'' = -A\cos t - B\sin t$

so $y'' + y = -A\cos t - B\sin t + A\cos t + B\sin t = 0$ ✓

(4) $y = A\cos\omega t + B\sin\omega t$ then $y'' = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$

$y'' + 16y = 0$ so $-\omega^2 A\cos\omega t - \omega^2 B\sin\omega t + 16A\cos\omega t + 16B\sin\omega t = 0$

$\Rightarrow A\cos\omega t(16-\omega^2) + B\sin\omega t(16-\omega^2) = (16-\omega^2)(A\cos\omega t + B\sin\omega t) = 0 = (16-\omega^2)y$

$y(0) = 2$ so $A = 2$, $y(\pi/8) = 3 \neq 0$ so $\omega = \pm 4$. If $\omega = 4$, then

$2\cos(\pi/2) + B\sin(\pi/2) = 3 \Rightarrow B = 3$ and if $\omega = -4$, then $B = -3$

So $(A, B, \omega) = (2, 3, 4)$ or $(2, -3, -4)$

⑧ $y''+4y=0 \quad \omega^2=4$ so if $\omega>0$, then $\omega=2$

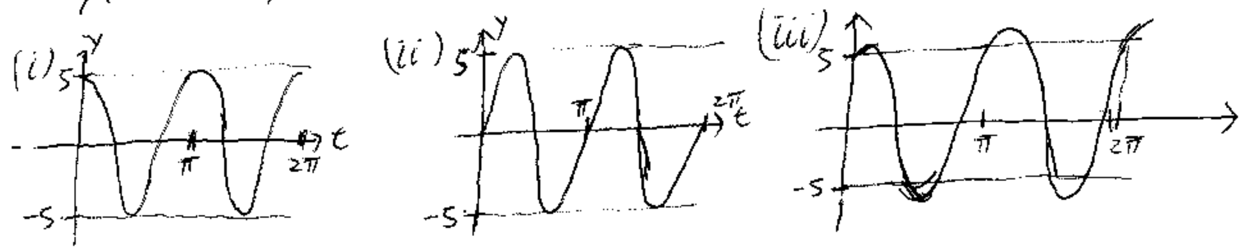
(a) So $y(t)=c_1\cos 2t + c_2 \sin 2t$ (for later, $y'=-2c_1\sin 2t + 2c_2\cos 2t$)

(b) (i) $y(0)=5 \quad y(0)=c_1\cos 0 = 5 \Rightarrow c_1=5$
$y'(0)=0 \quad y'(0)=2c_2\cos 0=0 \Rightarrow c_2=0$ $\quad \underline{y(t)=5\cos 2t}$

(ii) $y(0)=0 \quad y(0)=c_1\cos 0 = 0 \Rightarrow c_1=0$
$y'(0)=10 \quad y'(0)=2c_2\cos 0=10 \Rightarrow c_2=5$ $\quad \underline{y(t)=5\sin 2t}$

(iii) $y(0)=5 \Rightarrow c_1=5$
$y'(0)=5 \Rightarrow y'(0)=2c_2\cos 0=5 \Rightarrow c_2=2.5$ $\quad y(t)=5\cos(2t)+5/2\,\sin(2t)$

(c)



⑨ (a) III $\quad x(t)=\cos t$ $\quad$ (b) I $\quad x(t)=\sin(2t)$
or
II $\quad x(t)=-\sin t$

⑩ Large $\omega \Rightarrow$ smaller period of oscillation $\quad c_2=0$ since $s'(0)=0$ So Amplitude depends on c_1 (or just $s(0)$)

So (a) (iii) has the shortest period
(b) (iv) has the largest amplitude
(c) (iv) has the longest period
(d) (iii) has the largest maximum velocity ($V_{max}=A\cdot\omega$)

(14) $7\sin\omega t + 24\cos\omega t$ $\quad A = \sqrt{7^2+24^2} = \sqrt{625} = 25$

$\tan\varphi = \frac{c_1}{c_2} = \frac{7}{24} \Rightarrow \varphi = \tan^{-1}\frac{7}{24}$

So $7\sin\omega t + 24\cos\omega t = 25\sin(\omega t + \tan^{-1} 7/24)$

A. $\frac{dy}{dx} = y(y+1) \Rightarrow \frac{dy}{y(y+1)} = dx \Rightarrow \int \frac{1}{y}dy - \int\frac{1}{y+1}dy = \int dx$

$\frac{1}{y(y+1)} = \frac{1}{y} + \frac{-1}{y+1}$ $\quad \Rightarrow \ln|y| - \ln|y+1| = x + C = \ln\left|\frac{y}{y+1}\right|$

$\left|\frac{y}{y+1}\right| = e^{x+c} = e^x \cdot e^c$

$\frac{y}{y+1} = ke^x \quad (k = \pm e^c)$

$y = kye^x + ke^x$

$y = \frac{ke^x}{1-ke^x}$