Math 1b HW Solutions (Week 11)

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(2) $y''+4y'+4y=0 \qquad b=c=4$ so $b^2-4c=0$

So $y(t)=(c_1t+c_2)e^{-4t/2} = \underline{(c_1t+c_2)e^{-2t}}$

(6) $4z''+8z'+3z=0 \Rightarrow z''+2z'+3/4\,z=0 \qquad b^2-4c=4-4(3/4)=1>0$

So $z(t)=c_1e^{r_1t}+c_2e^{r_2t} \qquad r_1=-\frac{1}{2}(2)+\frac{1}{2}\sqrt{4-4(3/4)}=-1+\frac{1}{2}=-1/2$

$r_2=-\frac{1}{2}(2)-\frac{1}{2}\sqrt{1}=-3/2$

$=\underline{c_1e^{-1/2t}+c_2e^{-3/2t}}$

(7) $\frac{d^2x}{dt^2}+4\frac{dx}{dt}+8x=0 \qquad b=4, c=8 \quad b^2-4c=16-32=-16<0$

So $x(t)=c_1e^{\alpha t}\cos\beta t+c_2e^{\alpha t}\sin\beta t \qquad r=-\frac{1}{2}(4)\pm\frac{1}{2}\sqrt{-16}=-2\pm2i \Rightarrow \alpha=-2, \beta=2$

$x(t)=c_1e^{-2t}\cos 2t+c_2e^{-2t}\sin 2t$

(10) $y''+6y'+5y=0 \quad y(0)=5 \quad y'(0)=5 \qquad b^2-4c=36-20=16>0$

So $y(t)=c_1e^{r_1t}+c_2e^{r_2t} \qquad r_1=-\frac{1}{2}(6)+\frac{1}{2}\sqrt{16}=-1$

$r_2=-\frac{1}{2}(6)-\frac{1}{2}\sqrt{16}=-5$

$y(t)=c_1e^{-t}+c_2e^{-5t}$

$y(0)=c_1+c_2=5 \qquad y'(t)=c_1r_1e^{r_1t}+c_2r_2e^{r_2t}=-c_1e^{-t}-5c_2e^{-5t}$

$y'(0)=-c_1-5c_2=5$

$-4c_2=10 \Rightarrow c_2=-5/2 \Rightarrow c_1=15/2$

So $y(t)=\frac{15}{2}e^{-t}-\frac{5}{2}e^{-5t}$

(12) $y''+6y'+10y=0 \quad y(0)=0 \quad y'(0)=0 \qquad b^2-4c=36-40=-4<0$

So $y(t)=C_1e^{\alpha t}\cos\beta t+C_2e^{\alpha t}\sin\beta t \qquad r=-\frac{1}{2}(6)\pm\frac{1}{2}\sqrt{-4}=-3\pm2i \quad \alpha=-3,\ \beta=2$

$y'(t)=C_1[e^{\alpha t}\cdot(\sin\beta t)\beta+(\cos\beta t)\cdot\alpha e^{\alpha t}]+C_2[e^{\alpha t}\cos\beta t\cdot\beta+(\sin\beta t)\alpha e^{\alpha t}]$

$y(0)=C_1\cdot1\cdot1+C_2\cdot1\cdot0=0 \qquad y'(0)=0+C_2[1\cdot1\cdot(2)+0\cdot(-3)\cdot1]=0$

$\Rightarrow C_1=0 \qquad\qquad \Rightarrow 2C_2=0\Rightarrow C_2=0$

The only solution is $y=0$

(22) $s''+2\sqrt{2}\,s'+cs=0 \qquad b^2-4c=(2\sqrt{2})^2-4c=8-4c$

(a) $8-4c>0\Rightarrow 8>4c\Rightarrow 2>c$ (c) $8-4c=0\Rightarrow 8=4c\Rightarrow c=2$

(b) $8-4c<0\Rightarrow 8<4c\Rightarrow 2<c$

(24) $\frac{d^2y}{dt^2}-5\frac{dy}{dt}+ky=0 \qquad y=e^{2t}$ is a solution. $y'=2e^{2t} \quad y''=4e^{2t}$

So $4e^{2t}-5(2e^{2t})+k(e^{2t})=0\Rightarrow e^{2t}(4-10+k)=0$

e^{2t} is never 0 so $4-10+k=0\Rightarrow \underline{\underline{k=6}}$

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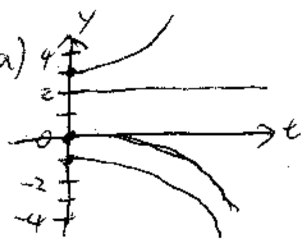
(1)(a)

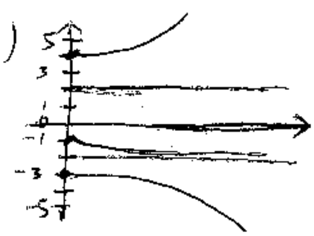


$y'=4y-8 \qquad y''=4y'=16y-32$

$y'>0$ when $y>2$ $\quad y''>0$ for $y>2$

$y'<0$ when $y<2$ $\quad y''<0$ for $y<2$

(b)



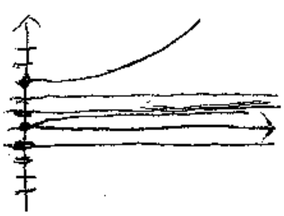
$y'=y^2-4 \qquad y''=2yy'-4=2y^3-8y-4$

$y'>0$ for $y<-2$ and $y>2$ $\qquad$ For $y<2$ (and positive), $y''<0$

$y'<0$ for $-2<y<2$

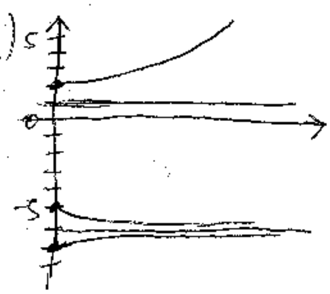
y'': $-$ | ≈-1.6 | $+$ | $\approx-.5$ | 0 | $-$ | ≈2.2 | $+$

(c)



$y'=(y-1)(y-2)(y+1)$

y': $-$ | -1 | $+$ | 1 | $-$ | 2 | $+$

(d) 

$y' = y^2 + 5y - 6 = (y+6)(y-1)$ $\quad y'' = 2y \cdot y' + 5y' = (2y+5)(y^2+5y-6)$

y': $+$ (−6) $-$ (1) $+$ $\quad$ y'': $-$ (−6) $+$ (−5/2) $-$ (1) $+$

(2) a. $y' = 0$ when $y = 3$ $\quad y' < 0$ for $y > 3$ and $y' > 0$ for $y < 3$ so $y' = 3 - y$ works here.
Notice that $y'' = -y' = y - 3$ which is negative for $y < 3$ and positive for $y > 3$

$\underline{y' = 3 - y}$

b. $\underline{y' = y - 3}$ works, $y' > 0$ for $y > 3$ and $y' < 0$ for $y < 3$ while $y' = 0$ when $y = 3$
Also $y'' = y' = y - 3$ is consistent with the graph ($y'' > 0$ for $y > 3$, $y'' < 0$ for $y < 3$)

c. There are two steady state solutions, $y = 2$ and $y = 0$. So try $y' = y(y-2)$
For $y > 2$, $y' > 0$ which is the opposite of what we want. So try $y' = y(2-y)$
This gives consistent values of y'. Check y''. $y'' = 2y' - 2y \cdot y' = 2y'(1-y)$
$= 2(1-y)(2-y) \cdot y = 2y(y-1)(y-2)$. This is also consistent with the graph.

$\underline{y' = y(2-y)}$

d. The two steady state solutions are $y = 1$ and $y = -1$ so try $\underline{y' = (y+1)(y-1)}$
$y'' = 2y \cdot y' = 2y(y+1)(y-1)$. This is good because $y' > 0$ for $y > 1$ and $y < -1$
and the y'' values are also consistent.

(3) a. The one equilibrium is at $y = 3$ and it's a stable equilibrium.

b. $y = 3$ is an unstable equilibrium and is the only equilibrium.

c. $y = 2$ is a stable equilibrium and $y = 0$ is an unstable equilibrium

d. $y = 1$ is an unstable equilibrium and $y = -1$ is a stable equilibrium.

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④ (a) $g(y)=0$ when $y=1, 3$ and 5 so the constant solutions are $y=1, 3$ or 5.

(b) To have an increasing solution, we want $\frac{dy}{dt}>0$ or $g(y)>0$

So $y(0)<1$, $3<y(0)<5$ will give an increasing function as a solution.

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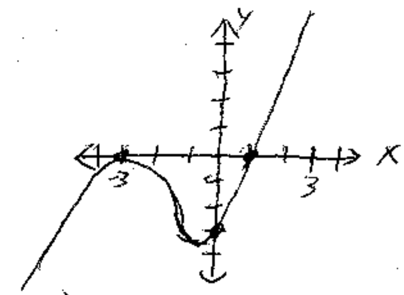
① The stable equilibria in figure 18 are the points on the y-axis (not the $\frac{dy}{dt}$-axis) where the arrows are pointing to. In this case, y_3.

A.

① $f(x)=(x+3)^2(x-1)$



There are roots at $x=1$ and $x=-3$. $x=-3$ is a double root so $f(x)$ is tangent to the x-axis there.

For large x, $f(x)$ is large positively (if x is positive)

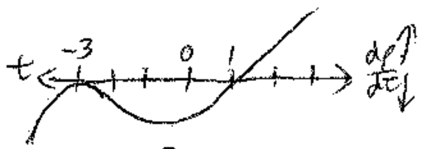
$f'(x)=(x+3)^2+2(x-1)(x+3)=(x+3)(3x+1)$

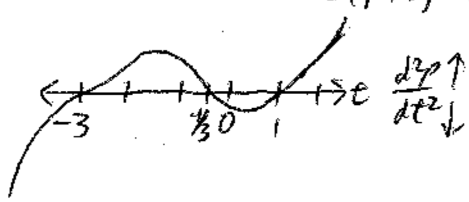
$f''(x)=2(x+3)+2(x-1)+2(x+3)$
$=6x+10$

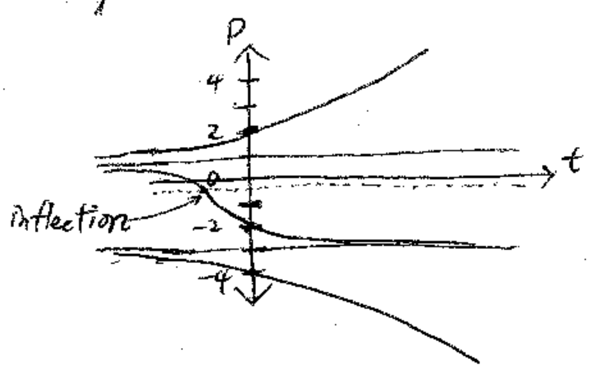
$f'(x)=0$ at $x=-3$ and $x=-1/3$

$f''(x)=0$ at $x=-5/4$

③ $\frac{dP}{dt}=(P+3)^2(P-1)$ $\quad$ $\frac{d^2P}{dt^2}=\left[(P+3)^2+2(P+3)(P-1)\right]\frac{dP}{dt}=(P+3)(P+3+2P-2)(P+3)^2(P-1)$
$=(P+3)^3(P-1)(3P+1)$







A_2

(4) For $a > 1$, P will be an increasing solution. It will also be increasing for $a < -3$ and $a > -1/3$.

(5)

$$\frac{dt}{dP} = \frac{1}{(P+3)^2(P-1)} = \frac{1}{16(P-1)} - \frac{P+7}{16(P+3)^2} = \frac{1}{16(P-1)} - \frac{1}{16(P+3)} - \frac{1}{4(P+3)^2}$$

$$\text{So } \int dt = \int\left(\frac{1}{16(P-1)} - \frac{P+7}{16(P+3)^2}\right)dP$$

$$t = \int \frac{1}{16(P-1)}\,dP - \int \frac{P+7}{16(P+3)^2}\,dP$$

$$= \frac{1}{16}\ln|P-1| - \int \frac{1}{16(P+3)}\,dP - \int \frac{1}{4(P+3)^2}\,dP$$

$$= \frac{1}{16}\ln|P-1| - \frac{1}{16}\ln|P+3| + \frac{1}{4}(P+3)^{-1} + P(0)$$

$$= \frac{1}{16}\ln\left|\frac{P-1}{P+3}\right| + \frac{1}{4(P+3)} + P(0)$$