A. $y''' = -y$

Let $y = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

1) $y(0) = 1 \Rightarrow a_0 = 1$; $y'(0) = 0 \Rightarrow a_1 = 0$; $y''(0) = 0 \Rightarrow a_2 = 0$

2) $y''' = \frac{3!}{0!} a_3 + \frac{4!}{1!} a_4 x + \frac{5!}{2!} a_5 x^2 + \frac{6!}{3!} a_6 x^3 + \frac{7!}{4!} a_7 x^4 + \cdots$

$-y = -a_0 - a_1 x - a_2 x^2 - a_3 x^3 - a_4 x^4 - \cdots$

$-a_0 = 3! a_3 \quad -a_1 = 4! a_4 \quad -a_2 = \frac{5!}{2!} a_5 \quad -a_3 = \frac{6!}{3!} a_6 \; \ldots \; -a_k = \frac{(k+3)!}{k!} a_{k+3}$

or $\frac{k!}{(k-3)!} a_k = -a_{k-3} \Rightarrow a_k = \frac{-(k-3)!}{k!} a_{k-3} \Rightarrow a_k = \frac{(k-6)!}{k!} a_{k-6}$

Notice that $a_1 = 0$ and $a_2 = 0$ so a_{3k+1} and a_{3k+2} are 0 for all k.

If k is a multiple of 3, then $a_k = \frac{(-1)^k}{k!}$ (the sign depends only on whether k is even or odd)

3) The series now is $y = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \cdots = \sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)!} (-1)^k$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ and $y < e^x$ term by term so y converges for all x.

4) $y = \frac{2}{3} e^{x/2} \cos\left(\frac{\sqrt{3}}{2} x\right) + \frac{1}{3} e^{-x}$. $y' = \frac{2}{3} e^{x/2} \cdot \left(-\frac{\sqrt{3}}{2}\right) \sin\left(\frac{\sqrt{3}}{2} x\right) + \cos\left(\frac{\sqrt{3}}{2} x\right) \cdot \frac{1}{3} e^{x/2} - \frac{e^{-x}}{3}$

$y'' = \frac{2}{3} e^{x/2} \left(-\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cos\left(\frac{\sqrt{3}}{2} x\right)\right) - \frac{\sqrt{3}}{2} \sin\left(\frac{\sqrt{3}}{2} x\right) \cdot \frac{1}{3} e^{x/2} + \cos\left(\frac{\sqrt{3}}{2} x\right) \cdot \frac{e^{x/2}}{6} - \frac{e^{x/2}}{3} \frac{\sqrt{3}}{2} \sin\left(\frac{\sqrt{3}}{2} x\right) + \frac{e^{-x}}{3}$

$= -\frac{e^{x/2}}{2} \cos\left(\frac{\sqrt{3}}{2} x\right) - \frac{\sqrt{3}}{6} e^{x/2} \sin\left(\frac{\sqrt{3}}{2} x\right) + \frac{e^{x/2}}{6} \cos\left(\frac{\sqrt{3}}{2} x\right) - \frac{\sqrt{3} e^{x/2}}{6} \sin\left(\frac{\sqrt{3}}{2} x\right) + \frac{e^{-x}}{3}$

$= -\frac{e^{x/2}}{3} \cos\left(\frac{\sqrt{3}}{2} x\right) - \frac{\sqrt{3}}{3} e^{x/2} \sin\left(\frac{\sqrt{3}}{2} x\right) + \frac{e^{-x}}{3}$

$y''' = +\frac{e^{x/2}}{3} \frac{\sqrt{3}}{2} \sin \frac{\sqrt{3}}{2} x - \cos\left(\frac{\sqrt{3}}{2} x\right) \frac{e^{x/2}}{6} - \frac{\sqrt{3}}{3} e^{x/2} \cos\left(\frac{\sqrt{3}}{2} x\right) \cdot \frac{\sqrt{3}}{2} - \sin\left(\frac{\sqrt{3}}{2} x\right) \frac{\sqrt{3}}{6} e^{x/2} - \frac{e^{-x}}{3}$

$= -\frac{2}{3} e^{x/2} \cos\left(\frac{\sqrt{3}}{2} x\right) - \frac{1}{3} e^{-x} = -y$ ✓

$y(0) = -(-2/3 - 1/3) = 1$ ✓

$y'(0) = 0 + 1/3 - 1/3 = 0$ ✓

$y''(0) = -1/3 - 0 + 1/3 = 0$ ✓

A=

5) Since $\frac{2}{3}e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)+\frac{1}{3}e^{-x}$ satisfies $y'''=-y$ and the solution to this equation is $\sum_{k=0}^{\infty}\frac{(-1)^k x^{3k}}{(3k)!}$, So $\sum_{k=0}^{\infty}\frac{(-1)^k x^{3k}}{3!}=\frac{2}{3}e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)+\frac{1}{3}e^{-x}$

This is the Maclaurin series.

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① $y'=xy$ $\quad y'=a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\dots$

$xy= \quad a_0x+a_1x^2+a_2x^3+a_3x^4+\dots$

$a_1=0 \Rightarrow a_{2k+1}=0$ for all k

$a_0=2a_2=8a_4=48a_6\dots$

$\quad 2 \quad 2\cdot4 \quad 2\cdot4\cdot6$

$a_{2k}=\frac{1}{(2k)(2k-2)\dots4\cdot2}a_0=\frac{1}{2(k)2(k-1)2(k-2)\dots2(2)2(1)}a_0$

$=\frac{1}{k!\,2^k}a_0$

But $y(0)=1$ so $a_0=1$

So $y=1+0+\frac{1}{2}x^2+0+\frac{1}{2\cdot4}x^4+\dots=\sum_{k=0}^{\infty}\frac{x^{2k}}{k!\,2^k}=\sum_{k=0}^{\infty}\frac{(x^2/2)^k}{k!}=e^{x^2/2}$

② $y'=a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\dots$

$x-y=-a_0+(1-a_1)x-a_2x^2-a_3x^3-a_4x^4-\dots$

$y(0)=0$ so $a_0=0 \Rightarrow a_1=0$ For ~~larger~~ k, $ka_k=-a_{k-1}$ so $a_k=\frac{-(k-1)!}{k!}a_{k-1}$

and recursively, we get $a_k=\frac{(2!)}{k!}a_2=\frac{2!}{k!}\frac{(1-a_1)}{2}=\frac{1-a_1}{k!}=\frac{(-1)^k}{k!}$ (by inspection)

So $y=0+0+\frac{1}{2}x^2-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\dots$

$=\left(1-x+\frac{1}{2}x^2-\frac{x^3}{3!}+\dots\right)-1+x$

$=e^{-x}+x-1$

(3) $y' = x\sin x \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + 7a_7 x^6 + \cdots$

$x\sin x = \qquad x^2 \qquad - \frac{1}{3!} x^4 \qquad + \frac{1}{5!} x^6 \cdots$

$y(0) = 1$ so $a_0 = 1, \quad a_1 = 0, \; a_2 = 0, \; a_3 = \frac{1}{3}, \; a_4 = 0, \; a_5 = \frac{-1}{5\cdot 3!}, \; a_7 = \frac{1}{7\cdot 5!}$

So $a_{2k} = 0$ for all k and $a_{2k+1} = \frac{(-1)^{k+1}}{(2k+1)(2k-1)!}$ for $k \geq 1$ $= \frac{(-1)^{k+1}\, 2k}{(2k+1)!}$

So $y = 1 + 0 + 0 + \frac{1}{3}x^3 + 0 - \frac{4}{5!}x^5 + 0 + \frac{6}{7!}x^7 + \cdots$

$= 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) - x + \frac{x^3}{2!} - \frac{x^5}{4!} + \frac{x^7}{6!} - \cdots$

$= 1 + \sin x - x\cos x$

By sep. of var

$\frac{dy}{dx} = x\sin x \qquad dy = x\sin x\, dx \Rightarrow y = \int x \sin x\, dx$

$y = -x\cos x + \int \cos x\, dx = \sin x - x\cos x + C$

$y(0) = 1$ so $C = 1 \quad \therefore y = 1 + \sin x - x\cos x.$

(4) $xy'' + y' + xy = 0$

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} \quad \text{so} \quad xy'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-1}$$

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1} = a_1 + \sum_{k=2}^{\infty} k a_k x^{k-1}$$

$$xy'' + y' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-1} + \sum_{k=2}^{\infty} k a_k x^{k-1} + a_1$$

From $y'(0) = 0$, we know that $a_1 = 0$

$$\text{so } xy'' + y' = \sum_{k=2}^{\infty} k^2 a_k x^{k-1} = -xy = -x \sum_{k=0}^{\infty} a_k x^k = -\sum_{k=0}^{\infty} a_k x^{k+1}$$

$$= -\sum_{j=2}^{\infty} a_{j-2} x^{j-1}$$

$\Rightarrow k^2 a_k = -a_{k-2}$, $a_1 = 0 \Rightarrow a_{2k+1} = 0$ for all k, $a_k = \frac{-1}{k^2} a_{k-2}$

$\therefore y = a_0 + 0 + \left(\frac{-1}{2^2} a_0\right) x^2 + 0 + \left(\frac{-1 \cdot -1}{4^2 \cdot 2^2} a_0\right) x^4 + \dots$

$\Rightarrow y = a_0 - \frac{1}{2(1!)} a_0 x^2 + \frac{1}{2^2(2!)^2} a_0 x^4 - \frac{1}{2^3(3!)^2} a_0 x^6 + \dots \quad a_0 = 1$

$$\text{so } y = 1 - \frac{1}{2(2!)} x^2 + \frac{1}{2^2(2!)^2} x^4 - \frac{1}{2^3(3!)^2} x^6 + \dots + \frac{(-1)^k}{2^k (k!)^2} x^{2k} + \dots$$