Math 1b HW Solutions (Week 2)

11.6

(12) $\sum_{k=1}^{\infty} \frac{4^k}{k^2}$ $\quad \lim_{k\to\infty} \frac{\frac{4^{k+1}}{(k+1)^2}}{\frac{4^k}{k^2}} = \lim_{k\to\infty} \frac{4k^2}{(k+1)^2} = 4 > 1$ so series diverges

(14) $\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k$ $\quad \lim_{k\to\infty} \frac{(k+1)(.5)^{k+1}}{k(.5)^k} = \lim_{k\to\infty} \frac{k+1}{2k} = \frac{1}{2} < 1$ so series converges

(16) $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$ $\quad \lim_{k\to\infty} \frac{\frac{k+1}{(k+1)^2+1}}{\frac{k}{k^2+1}} = \lim_{k\to\infty} \frac{(k+1)(k^2+1)}{k[(k+1)^2+1]} = 1$ The test is inconclusive ~~0 < 1 so series converges~~

(18) $\sum_{k=1}^{\infty} \left(\frac{k}{100}\right)^k$ $\quad \lim_{k\to\infty} \left[\left(\frac{k}{100}\right)^k\right]^{1/k} = \lim_{k\to\infty} \frac{k}{100} = \infty$ so series diverges

(26) $\sum_{k=1}^{\infty} \frac{k^2}{k^3+1} = a_n$ Compare with $\sum_{k=1}^{\infty} \frac{1}{k} = b_n$ $\quad \lim_{k\to\infty} \frac{\frac{k^2}{k^3+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^3}{k^3+1} = 1 < \infty$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, so does $\sum_{k=1}^{\infty} \frac{k^2}{k^3+1}$

(30) $\sum_{k=1}^{\infty} \frac{2+(-1)^k}{5^k} = \underbrace{\sum_{k=1}^{\infty} \frac{2}{5^k}}_{A} + \underbrace{\sum_{k=1}^{\infty} \left(\frac{-1}{5}\right)^k}_{B}$

A converges because it's a geometric series with $a = 2/5$ $r = 1/5 < 1$

B converges because it's a geometric series with $a = -1/5$ $r = -1/5$ $(|r| < 1)$

OR $\frac{2+(-1)^k}{5^k} < \frac{3}{5^k}$ for all k and $\sum_{k=1}^{\infty} \frac{3}{5^k}$ converges (geometric series)

So $\sum_{k=1}^{\infty} \frac{2+(-1)^k}{5^k}$ converges

(44) $\sum_{k=1}^{\infty} \frac{(k!)^2(2^k)}{(2k+2)!}$ Use ratio test $\lim_{k\to\infty} \frac{[(k+1)!]^2\, 2^{k+1}}{(2k+4)!} \cdot \frac{(2k+2)!}{(k!)^2\, 2^k} = \lim_{k\to\infty} \frac{2(k+1)^2}{(2k+4)(2k+3)} = \frac{1}{2}$

$1/2 < 1$ so series converges.

(46) $1 + \frac{1\cdot 3}{3!} + \frac{1\cdot 3\cdot 5}{5!} + \dots$ The n^{th} term is $\frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n-1)}{(2n-1)!}$

Using the ratio test, $\lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n+1)}{(2n+1)!} \cdot \frac{(2n-1)!}{1\cdot 3\cdot 5 \cdots (2n-1)} = \lim_{n\to\infty} \frac{\cancel{2n+1}}{(\cancel{2n+1})(2n)} = \lim_{n\to\infty} \frac{1}{2n} = 0$

So series converges

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(5) $\sum_{k=1}^{\infty}(-1)^{k+1}e^{-k}$ Alt. series test: $\lim_{k\to\infty} e^{-k}=0$ and $|a_k|<|a_{k-1}|$ for all k

So series converges

(8) $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2^k}{k!}$ Ratio test: $\lim_{k\to\infty}\frac{2^{k+1}}{(k+1)!}\cdot\frac{k!}{2^k}=\lim_{k\to\infty}\frac{2}{k+1}=0$ so series absolutely converges

(10) $\sum_{k=1}^{\infty}(-1)^{k}\frac{k}{5^k}$ Ratio test: $\lim_{k\to\infty}\frac{k+1}{5^{k+1}}\cdot\frac{5^k}{k}=\lim_{k\to\infty}\frac{k+1}{5k}=1/5<1$ so series absolutely converges

(22) $\sum_{k=1}^{\infty}\frac{|\sin k|}{k^3}\le\sum_{k=1}^{\infty}\frac{1}{k^3}$ which converges by the p-series test so $\sum_{k=1}^{\infty}\frac{\sin k}{k^3}$ converges absolutely

(28) $\sum_{k=1}^{\infty}\frac{k\cos(k\pi)}{k^2+1}=\sum_{k=1}^{\infty}\frac{(-1)^k\cdot k}{k^2+1}$ converges using the alt. series test but

limit comparison test

$\sum_{k=1}^{\infty}\frac{k}{k^2+1}$ diverges because $\lim_{k\to\infty}\frac{\frac{k}{k^2+1}}{\frac{1}{k}}=1$ and $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges.

So $\sum_{k=1}^{\infty}\frac{k\cos(k\pi)}{k^2+1}$ converges conditionally

(48) $\frac{\pi^4}{90}=1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\dots=\sum_{k=1}^{\infty}\frac{1}{k^4}=\underbrace{\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\dots\right)}_{A}+\left(\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\dots\right)$

$=A+\frac{1}{2^4}\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right)=A+\frac{1}{16}\sum_{k=1}^{\infty}\frac{1}{k^4}=A+\frac{\pi^4}{90\cdot16}$

So $A=\frac{\pi^4}{90}-\frac{\pi^4}{90\cdot16}=\frac{\pi^4}{96}$

A: Find all x for which $\sum_{n=1}^{\infty}n\left(\frac{x}{5x+6}\right)^n$ converges

Let $r=\frac{x}{5x+6}$. Then using the ratio test, the series converges when $\left|\lim_{n\to\infty}\frac{(n+1)r^{n+1}}{nr^n}\right|<1$ which is when $|r|<1$. If $5x+6>0$, then $x<5x+6\Rightarrow -3/2<x$ (only true when $x>-6/5>-3/2$) or $x>-5x-6\Rightarrow x>-1$. If $5x+6<0$, then $x>5x+6\Rightarrow -3/2>x$ or $-5x-6>x\Rightarrow -1>x$ (which is already true because $-3/2>x$). So the series will surely converge for $x>-1$ and $x<-3/2$. Now, check the endpoints. If $x=-1$, then $r=-1$. If $x=-3/2$ then $r=1$. These correspond to the series $\sum_{n=1}^{\infty}n\cdot(-1)^n$ and $\sum_{n=1}^{\infty}n$ respectively. The first series oscillates (hence it doesn't converge) and the second series clearly diverges ($\lim_{n\to\infty}n=\infty\neq0$)

So the values of x for which the series converges are: $x>-1$ or $x<-3/2$