Math 1b HW Solutions (Week 3)

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(5) (a) The domain of $f(x)$ is the values of x for which the series $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{k-1}}{2^{k-1}}$ converges

$$\left|\lim_{k\to\infty}\frac{x^k}{2^k}\cdot\frac{2^{k-1}}{x^{k-1}}\right|=\left|\frac{x}{2}\right|<1 \iff -2<x<2$$

Note that $x=2$ and $x=-2$ don't work because $\sum_{k=1}^{\infty}(-1)^{k-1}$ does not converge.

Domain $=(-2,2)$

(b) Clearly $f(0)=1$, $f(1)=\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{1}{2}\right)^{k-1}$ = geometric series w/ $a=1$, $r=-1/2$ $=\frac{1}{1+1/2}=2/3$

(6) (a) Similar to #5, $f(x)=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(x-5)^{k-1}}{3^{k-1}}$, $\left|\lim_{k\to\infty}\frac{(x-5)^k}{3^k}\cdot\frac{3^{k-1}}{(x-5)^{k-1}}\right|=\left|\frac{x-5}{3}\right|<1$

which means the radius of convergence is 3 and the range of x is $2<x<8$

Again, $x=2$ and $x=8$ do not work because $\sum_{k=1}^{\infty}(-1)^{k-1}$ does not converge

Domain $=(2,8)$

(b) In general, $f(x)$ is a geometric series with $a=1$, $r=\frac{(x-5)}{-3}$ and the sum of the series is $\frac{1}{1+\frac{(x-5)}{3}}=\frac{3}{x-2}$ where $x\in(2,8)$, so $f(3)=3$ and $f(6)=3/4$

(8) $\sum_{k=0}^{\infty}3^k x^k$ Using the ratio test $\lim_{k\to\infty}\left|\frac{(3x)^{k+1}}{(3x)^k}\right|=|3x|<1 \iff -1/3<x<1/3$

So the radius of convergence is $1/3$

Notice that $x=\pm 1/3$ does not work because $\sum_{k=0}^{\infty}1$ and $\sum_{k=0}^{\infty}(-1)^k$ do not converge

So the interval of convergence is $(-1/3, 1/3)$

(10) $\sum_{k=0}^{\infty}\frac{k!}{2^k}x^k$ Using the ratio test $\left|\lim_{k\to\infty}\frac{(k+1)!\,x^{k+1}}{2^{k+1}}\cdot\frac{2^k}{k!\,x^k}\right|=\left|\lim_{k\to\infty}\frac{x(k+1)}{2}\right|=\infty$

So the radius of convergence is 0 and the interval of convergence is $x=0$.

(14) $\sum_{k=0}^{\infty}\frac{(-2)^k x^{k+1}}{k+1}$ $\left|\lim_{k\to\infty}\frac{(-2)^{k}x^{k+1}}{k+1}\cdot\frac{k}{(-2)^{k-1}\cdot x^k}\right|=\left|\lim_{k\to\infty}\frac{-2xk}{k+1}\right|=|-2x|=2|x|<1\Rightarrow -1/2<x<1/2$

So the radius of convergence is $1/2$,

If $x=-1/2$, the series becomes $\sum_{k=0}^{\infty}\frac{1}{2(k+1)}$ which diverges (use the limit comparison test with $\sum_{k=1}^{\infty}1/k$)

On the other hand, if $x=1/2$, the series becomes $\sum_{k=0}^{\infty}\frac{(-1)^k}{2(k+1)}$ which converges by the alternating series test. So the interval of convergence is $(-1/2, 1/2]$

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(18) $\sum_{k=1}^{\infty} (-1)^k \frac{x^{3k}}{k^{3/2}}$ $\quad \left| \lim_{k\to\infty} \frac{x^{3k+3}}{(k+1)^{3/2}} \cdot \frac{k^{3/2}}{x^{3k}} \right| = |x^3| < 1 \Rightarrow -1 < x < 1$

So the radius of convergence is 1.

Note that the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{3/2}}$ converges absolutely by the p-series test.

So the interval of convergence is $[-1, 1]$

(30) $\sum_{k=0}^{\infty} \frac{(2x-3)^k}{4^{2k}}$ $\quad \left| \lim_{k\to\infty} \frac{(2x-3)^{k+1}}{4^{2k+2}} \cdot \frac{4^{2k}}{(2x-3)^k} \right| = \frac{1}{16}|2x-3| < 1 \Leftrightarrow -13/2 < x < 19/2$

So the radius of convergence is 8,

When $x = 19/2$ or $-13/2$, the series becomes $\sum_{k=0}^{\infty} 1$ and $\sum_{k=0}^{\infty} (-1)^k$ respectively which both diverge. So the interval of convergence is $(-13/2, 19/2)$

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(1) (a) $f^{(k)}(x) = (-1)^k e^{-x}$ so $e^{-x} \approx 1 - x + \frac{x^2}{2}$ (quadratic)

$f^{(k)}(0) = (-1)^k$ $\quad \approx 1 - x$ (linear)

$\boxed{\text{In General } f(x) \approx f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \dots}$

(b) $f(x) = \cos x$ $\quad f'(x) = -\sin x$ $\quad f''(x) = -\cos x$ $\quad f^{(3)}(x) = \sin x$ $\quad x_0 = 0$

$\cos x \approx 1 - \frac{x^2}{2}$ (quadratic) $\quad \cos x \approx 1$ (linear)

(c) $f(x) = \sin x$ $\quad f'(x) = \cos x$ $\quad f''(x) = -\sin x$ $\quad x_0 = \pi/2$

$\sin x \approx 1 - \frac{(x-\pi/2)^2}{2}$ (quadratic) $\quad \sin x \approx 1$ (linear)

(d) $f(x) = \sqrt{x}$ $\quad f'(x) = \frac{1}{2\sqrt{x}}$ $\quad f''(x) = \frac{-1}{4(x)^{3/2}}$ $\quad x_0 = 1$

$\sqrt{x} \approx 1 + \frac{(x-1)}{2} - \frac{(x-1)^2}{8}$ (quadratic) $\quad \sqrt{x} \approx 1 + \frac{(x-1)}{2}$ (linear)

(7) Let p_n be the n^{th} order Maclaurin polynomial, In general $p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots \frac{f^{(n)}(0)}{n!}x^n$

$f(x) = e^{-x}$ $\quad p_0(x) = 1$ $\quad p_1(x) = 1 - x$ $\quad p_2(x) = 1 - x + \frac{x^2}{2!}$ $\quad p_3(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$

$p_4(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$ $\quad p(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$

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(10) $f(x)=\sin \pi x \quad f'(x)=\pi \cos \pi x \quad f''(x)=-\pi^2 \sin \pi x$ so $f^{(k)}(0)=0$ when k is even and alternates between π^k and $-\pi^k$ when k is odd.

$p_0(x)=0 \quad p_1(x)=\pi x \quad p_2(x)=\pi x \quad p_3(x)=\pi x-\frac{\pi^3 x^3}{3!}$

$p_4(x)=\pi x-\frac{\pi^3 x^3}{3!} \quad p(x)=\sum_{k=1}^{\infty} \frac{\pi^{(2k-1)} x^{(2k-1)}}{(-1)^{k+1}(2k-1)!} x^{2k-1}=\sum_{k=0}^{\infty}(-1)^k \frac{\pi^{(2k+1)} x^{2k+1}}{(2k+1)!}$

(12) $f(x)=\frac{1}{1+x}=(1+x)^{-1} \quad f^{(k)}(x)=(-1)^k \frac{k!}{(1+x)^{k+1}} \quad f^{(k)}(0)=(-1)^k k!$

$p_0(x)=1 \quad p_1(x)=1-x \quad p_2(x)=1-x+x^2 \quad p_3(x)=1-x+x^2-x^3$ ~~$p_4(x)$~~

$p_4(x)=1-x+x^2-x^3+x^4 \quad p(x)=\sum_{k=0}^{\infty}(-1)^k x^k$

(16) $f(x)=xe^x \quad f^{(k)}(x)=(k+x)e^x \quad f^{(k)}(0)=k$

$p_0(x)=0 \quad p_1(x)=x \quad p_2(x)=x+x^2 \quad p_3(x)=x+x^2+\frac{x^3}{2!} \quad p_4(x)=x+x^2+\frac{x^3}{2!}+\frac{x^4}{3!}$

$p(x)=\sum_{k=1}^{\infty} \frac{x^k}{(k-1)!}$

(22) $f(x)=\frac{1}{x+2} \quad f^{(k)}(x)=\frac{(-1)^k k!}{(x+2)^{k+1}} \quad f^{(k)}(3)=\frac{(-1)^k k!}{5^{k+1}}$ Let t_n be the n^{th} taylor polynomial

$t_0(x)=\frac{1}{5} \quad t_1(x)=\frac{1}{5}-\frac{(x-3)}{25} \quad t_2(x)=\frac{1}{5}-\frac{(x-3)}{25}+\frac{(x-3)^2}{125}$

$t_3(x)=\frac{1}{5}-\frac{(x-3)}{25}+\frac{(x-3)^2}{125}-\frac{(x-3)^3}{625} \quad t_4(x)=\frac{1}{5}-\frac{(x-3)}{25}+\frac{(x-3)^2}{125}-\frac{(x-3)^3}{625}+\frac{(x-3)^4}{3125}$

$t(x)=\sum_{k=0}^{\infty} \frac{(-1)^k (x-3)^k}{5^{k+1}}=\sum_{k=0}^{\infty} \frac{(-1)(x-3)^k}{5^{k+1}}$

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(24) $f(x) = \cos x \quad x_0 = \frac{\pi}{2} \quad f^{(k)}(\frac{\pi}{2}) = 0$ if k is even, -1 and $+1$ alternately if k is odd.

$t_0(x) = 0 \quad t_1(x) = -(x - \pi/2) \quad t_2(x) = -(x - \pi/2) \quad t_3(x) = -(x - \pi/2) + \frac{(x-\pi/2)^3}{3!}$

$t_4(x) = -(x - \pi/2) + \frac{(x-\pi/2)^3}{3!} \quad t(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} (x - \pi/2)^{2k+1}$

A: If $\sum a_n x^n$ has radius of convergence 5, what is the radius of convergence of $\sum a_n x^{2n}$?

$\sum a_n x^{2n} = \sum a_n (x^2)^n$ Since $\sum a_n x^n$ has radius of convergence 5,

$|x| < 5$. So for $\sum a_n x^{2n}$ to converge, $|x^2| < 5$ which means $|x| < \sqrt{5}$

so the radius of convergence is $\sqrt{5}$.