Math 1b HW Solutions (Week 9)

9.8

① (a) Improper because there is an infinite discontinuity at $x=3$

(b) NOT Improper because $\frac{1}{x+3}$ is continuous over $[1,5]$

(c) Improper because there is an infinite discontinuity at $x=0$

(d) Improper because of an infinite interval of integration.

(e) Improper because of an infinite interval of integration and an infinite discontinuity at $x=1$

(f) NOT Improper because $\tan x$ is continuous over $[0, \pi/4]$

④ $\int_{-1}^{\infty} \frac{x}{1+x^2}dx \to \lim_{a\to\infty} \int_{-1}^{a} \frac{x}{1+x^2}dx = \lim_{a\to\infty} \frac{1}{2}\ln(1+x^2)\Big|_{-1}^{a} = \lim_{a\to\infty} \frac{1}{2}\ln(1+a^2) - \frac{1}{2}\ln(2)$

$= \infty$, divergent

⑥ $\int_{0}^{\infty} xe^{-x^2}dx \to \lim_{l\to\infty} \int_{0}^{l} xe^{-x^2}dx = \lim_{l\to\infty} -\frac{1}{2}e^{-x^2}\Big|_{0}^{l} = \lim_{l\to\infty} -\frac{1}{2}\left[e^{-l^2} - e^{0}\right] = \frac{1}{2}$

⑩ $\int_{-\infty}^{2} \frac{dx}{x^2+4} \to \lim_{l\to-\infty} \int_{l}^{2} \frac{dx}{x^2+4} = \lim_{l\to\infty} \frac{1}{2}\tan^{-1}\frac{x}{2}\Big|_{l}^{2} = \lim_{l\to\infty} \frac{1}{2}(\tan^{-1}1 - \tan^{-1} l/2)$

$= \frac{1}{2}(\pi/4 + \pi/2) = \frac{3\pi}{8}$

⑭ $\int_{-\infty}^{\infty} \frac{x\,dx}{\sqrt{x^2+2}} = \int_{-\infty}^{0} \frac{x}{\sqrt{x^2+2}}dx + \int_{0}^{\infty} \frac{x}{\sqrt{x^2+2}}dx$, Look at $\int_{0}^{\infty} \frac{x}{\sqrt{x^2+2}}dx$

$\int_{0}^{\infty} \frac{x}{\sqrt{x^2+2}}dx = \lim_{a\to\infty} \sqrt{x^2+2}\Big|_{0}^{a} = \infty$ So the whole integral is divergent.

(16) $\int_{-\infty}^{\infty} \frac{e^{-t}}{1+e^{-2t}}dt = \int_{-\infty}^{0} \frac{e^{-t}}{1+e^{-2t}}dt + \int_{0}^{\infty} \frac{e^{-t}}{1+e^{-2t}}dt$

$$\int_{-\infty}^{0} \frac{e^{-t}}{1+e^{-2t}}dt = \lim_{a\to-\infty} -\tan^{-1}(e^{-t})\Big|_a^0 = \lim_{a\to-\infty} -\tan^{-1}(1)+\tan^{-1}(e^{-a}) = -\frac{\pi}{4}+\frac{\pi}{2} = \frac{\pi}{4}$$

$$\int_{0}^{\infty} \frac{e^{-t}}{1+e^{-2t}}dt = \lim_{l\to\infty} \int_0^l \frac{e^{-t}}{1+e^{-2t}}dt = \lim_{l\to\infty} -\tan^{-1}(e^{-t})\Big|_0^l = 0+\frac{\pi}{4} = \frac{\pi}{4}$$

So $\int_{-\infty}^{\infty} \frac{e^{-t}}{1+e^{-2t}}dt = \frac{\pi}{4}+\frac{\pi}{4} = \frac{\pi}{2}$

(18) $\int_0^8 (\sqrt[3]{x})^{-1}dx = \int_0^8 x^{-1/3}dx = \lim_{a\to 0^+} \frac{3x^{2/3}}{2}\Big|_a^8 = \frac{3}{2}(4-a^{2/3}) = 6$

(26) $\int_{-2}^{2} \frac{dx}{x^2} = 2\int_0^2 \frac{dx}{x^2}$ by symmetry

$$2\int_0^2 \frac{dx}{x^2} = 2\cdot \lim_{l\to 0^+}\int_l^2 \frac{dx}{x^2} = 2\cdot \lim_{l\to 0^+} -x^{-1}\Big|_l^2 = \lim_{l\to 0^+} 2\cdot\left(-\tfrac{1}{2}+\tfrac{1}{l}\right) = \infty$$ divergent

(49) Volume $= \lim_{l\to\infty} \int_1^l \pi\left(\frac{1}{x}\right)^2 dx = \pi \lim_{l\to\infty} -x^{-1}\Big|_1^l = \lim_{l\to\infty}\left(\frac{-1}{l}+\frac{1}{1}\right)\cdot\pi = \pi$

Area $= \lim_{l\to\infty} \int_1^l 2\pi\left(\frac{1}{x}\right)\sqrt{1+\left(\frac{1}{x^2}\right)^2}\, dx$

Notice that $\sqrt{1+\frac{1}{x^4}} > 1$ for all x in $[1,\infty)$ so $2\pi\left(\frac{1}{x}\right)\sqrt{1+\frac{1}{x^4}} > \frac{2\pi}{x}$ for all x in $[1,\infty)$

But $\int_1^{\infty} \frac{2\pi}{x}dx = {}"2\pi[\ln|\infty| - \ln(1)]" = \infty$ so the area is infinite.

(52) (a) $\int_0^{\pi} \sin x\, dx = 1$ and $\int_{\pi}^{2\pi} \sin x\, dx = -1$ In fact, $\int_{2n\pi}^{(2n+1)\pi} \sin x\, dx = 1$

and $\int_{(2n+1)\pi}^{(2n+2)\pi} \sin x\, dx = -1$ so $\int_0^{\infty} \sin x\, dx = \sum_{n=0}^{\infty} \int_{2n\pi}^{(2n+1)\pi} \sin x\, dx + \int_{(2n+1)\pi}^{(2n+2)\pi} \sin x\, dx$

$= 1-1+1-1+\ldots$ which is an oscillating series so it diverges.

$\int_0^{\infty} \cos x\, dx$ can be looked at in the same way except with an initial shift value of $1/2$

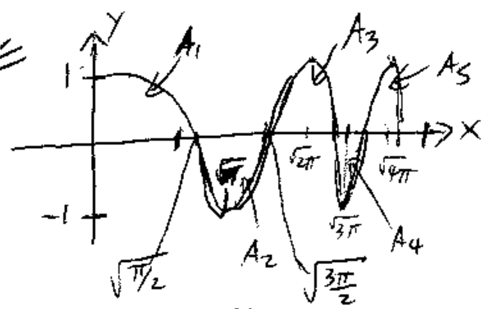
(55) weight $= \frac{k}{x^2}$ where $x =$ dist. from center of earth, weight at 4000 mi $= 6000$ lbs.

(a) $6000 = \frac{k}{(4000)^2}$ so $k = 9.6 \times 10^{10}$ and

$$W = \int_{4000}^{4000+\ell} 9.6 \times 10^{10} \cdot x^{-2}\,dx \text{ mi-lb}$$

(b) $\int_{4000}^{\infty} 9.6 \times 10^{10} x^{-2}\,dx = 9.6 \times 10^{10} \cdot \lim_{a \to \infty} -x^{-1}\Big|_{4000}^{a} = 9.6 \times 10^{10} \cdot \left(0 + \frac{1}{4000}\right)$ mi-lb $= 2.4 \times 10^{7}$ mi-lb

A.



Notice that $|A_1| > |A_2| > \dots$

and that $\int_0^\infty \cos(x^2)\,dx = \sum_{n=1}^{\infty} A_n$

The graph crosses the x-axis when $x = \sqrt{\frac{(2n+1)\pi}{2}}$ for $n = \{0, 1, \dots\}$.

So if $\sum_{n=1}^{\infty} A_n$ converges, so does the integral. From the graph,

clearly $|A_{i+1}| < |A_i|$ and the signs of each A_n alternate by setup.

So by the alternating series test, $\sum_{n=1}^{\infty} A_n$ converges and so

$\int_0^\infty \cos(x^2)\,dx$ is convergent.