Math 1b Homework Solutions (Week 4)

11.10

6 (a) $\cos \alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots$ so $\cos 2x = 1 - \frac{4x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + \cdots$

The radius of convergence is still $+\infty$

(b) $x^2e^x = x^2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdots$ The radius of convergence is still $+\infty$

(c) $xe^{-x} = x\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) = x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \cdots$ The radius of convergence is still $+\infty$

(d) $\sin x^2 = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$ The radius of convergence is still $+\infty$

16 (a) $\frac{\tan^{-1}x}{(1+x)} = \frac{(x - x^3/3 + x^5/5 - x^7/7 + \cdots)}{(1+x)} = \left(x - x^2 + \frac{2}{3}x^3 - \frac{2}{3}x^4 \cdots\right)$

$$\begin{array}{r l}
 & x - x^2 + \tfrac{2}{3}x^3 - \tfrac{2}{3}x^4 \cdots \\
1+x \,\big) & x + 0x^2 - x^3/3 + 0x^4 + x^5/5 \\
 & \underline{x + x^2} \\
 & -x^2 - x^3/3 \\
 & \underline{-x^2 - x^3} \\
 & \tfrac{2}{3}x^3 + 0x^4 \\
 & \underline{\tfrac{2}{3}x^3 + \tfrac{2}{3}x^4} \\
 & -\tfrac{2}{3}x^4 + x^5/5
\end{array}$$

(b) $\frac{\ln(1+x)}{(1-x)} = \frac{(x - x^2/2 + x^3/3 - x^4/4 + \cdots)}{(1-x)} = x + \frac{1}{2}x^2 + \frac{5}{6}x^3 + \frac{7}{12}x^4 + \cdots$

26 $x^2\cos 2x = x^2 - \frac{2^2x^4}{2!} + \frac{2^4x^6}{4!} - \frac{2^6x^8}{6!} + \cdots$ ($x^2 \cdot$ series from 6(a))

$= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} x^{2k+2}}{(2k)!} = f(x)$ But $f^{(99)}(x) = cx^3 + \{$something with $x)$

which means $f^{(99)}(0) = 0$

(33) (a) $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \quad -1 < x < 1$

$\left(\frac{1}{1-x}\right)' = \frac{+1}{(1-x)^2} = \sum_{k=0}^{\infty} kx^{k-1}$ so $\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}$ 1 will do because the first term is 0 if $k=0$.

(b) $\int \frac{1}{1-x}\,dx = -\ln(1-x) + C$

$\int \frac{1}{1-x}\,dx = \int \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = \sum_{k=1}^{\infty} \frac{x^k}{k} + C$ $\quad$ $\ln 1 = 0$ so $C = 0$

$+C \qquad +C$

and $-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$ and $-1<x<1$

(34) (a) $\sum_{k=1}^{\infty} \frac{k}{3^k} = \sum_{k=1}^{\infty} k \cdot \left(\frac{1}{3}\right)^k = \frac{1/3}{(1-1/3)^2} = 3/4$

(b) $\sum_{k=1}^{\infty} \frac{1}{k(4)^k} = \sum_{k=1}^{\infty} \frac{(1/4)^k}{k} = -\ln(1-1/4) = -\ln(3/4) = \ln(4/3)$

A. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!} = f'(x)$$

$$\text{so } f(x) = \int \frac{\sin x}{x}\,dx = \sum_{k=0}^{\infty} \int (-1)^k \frac{x^{2k}}{(2k+1)!} + C$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)(2k+1)!} + C$$

$f(0) = 1$ so $C = 1$

So the series is $1 + \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)(2k+1)!}$

The radius of convergence is $+\infty$ because $\frac{x^{2k+1}}{(2k+1)(2k+1)!} \le \frac{x^{2k+1}}{(2k+1)!}$ for all x.

and $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$ converges because ~~the~~ the series for $\sin x$ converges absolutely.

11.9

(3) $\sqrt{e} = e^{0.5} = 1 + 0.5 + \frac{(0.5)^2}{2!} + \frac{(0.5)^3}{3!} + \frac{(0.5)^4}{4!} + \frac{(0.5)^5}{5!} + \cdots$

≈ 1.6487 (by calculator)

$|R_n(0.5)| \le 0.00005$ is what we want so...

$|R_n(0.5)| \le \frac{\sqrt{e}}{(n+1)!}(\cancel{1/2})^{n+1} \le \frac{2}{(n+1)!}(.5)^{n+1} \le 0.0005$

This is true when $n \ge 5$ so approximate $e^{0.5}$ as

$1 + 0.5 + 0.125 + .02083 + .0026041\overline{6} \approx 1.6484$

(4) $1/e = e^{-1}$ Check that for $-1 \le x \le 0$, $f^{(k)}(x) = e^x \le 1$

so $|R_n(-1)| \le \frac{1}{(n+1)!} \cdot 1^{n+1} = \frac{1}{(n+1)!} \le 0.0005 \Rightarrow n \ge 6$

hence $e^{-1} \approx 1 - 1 + \frac{1}{2} - 1/3! + 1/4! - 1/5! + 1/6! \approx 0.3681$

(6) $\tan^{-1} 0.1 = (0.1) - \frac{(0.1)^3}{3} + \frac{(0.1)^5}{5} - \frac{(0.1)^7}{7} + \cdots$

but since $\frac{(0.1)^3}{3} < 0.5 \times 10^{-3} = 0.0005$, the approximation

$\tan^{-1} 0.1 \approx 0.1$ is sufficient to 3 decimal places

≈ 0.100

B. $\ln((1+2x)(1+5x)) = \ln(1+2x) + \ln(1+5x)$

So long as the Maclaurin series for each term converges, we can rearrange and group the terms as we deem fit.

$\ln(1+2x) = (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \cdots$ for $-1 < 2x \le 1 \Leftrightarrow -1/2 < x \le 1/2$

$\ln(1+5x) = (5x) - \frac{(5x)^2}{2} + \frac{(5x)^3}{3} - \frac{(5x)^4}{4} + \cdots$ for $-1/5 < x \le 1/5$

So if $-1/5 < x \le 1/5$ (when both series converge), we can add the series.

$\ln((1+2x)(1+5x)) = \ln(1+2x) + \ln(1+5x) = (2x+5x) - \frac{(2x)^2+(5x)^2}{2} + \frac{(2x)^3+(5x)^3}{3} - \cdots$

$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{((2x)^k + (5x)^k)}{k}$ for $-1/5 < x \le 1/5$