math 1b HW solutions
§ 11.10-12

§ 11.10

(4), (12), 14, 26, 36, 40, (45), 54, (56)

14.// Find Taylor series for $f(x)=\sqrt{x}$ centered at $x=4$

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$x^{1/2}$	2
1	$\frac{1}{2}x^{-1/2}$	2^{-2}
2	$-\frac{1}{4}x^{-3/2}$	-2^{-5}
3	$\frac{3}{8}x^{-5/2}$	$3\cdot 2^{-8}$
4	$-\frac{15}{16}x^{-7/2}$	$-15\cdot 2^{-11}$
...	...	...

Notice we can write $f^{(n)}(4)$ as:

$$=\frac{(-1)^{n-1}\, 1\cdot 3\cdot 5\cdot \dots \cdot(2n-3)}{2^{3n-1}} \quad \text{for } n\geq 2$$

$$\text{Taylor: } f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\dots$$

$$\boxed{\begin{gathered}\sqrt{x}=2+\tfrac{1}{4}(x-4)-\tfrac{1}{64}(x-4)^2+\tfrac{1}{512}(x-4)^3-\dots\\ \text{OR}\\ \sqrt{x}=2+\tfrac{1}{4}(x-4)+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\,1\cdot 3\cdot 5\cdot \dots \cdot(2n-3)}{2^{3n-1}\cdot n!}(x-4)^n\end{gathered}}$$

26.// $f(x)=x\cos 2x$

We know $\cos x = 1-x^2/2!+x^4/4!-x^6/6!+\dots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$

$$\cos 2x = 1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\dots = \sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!}$$

$$x\cos 2x = x-\frac{x(2x)^2}{2!}+\frac{x(2x)^4}{4!}-\frac{x(2x)^6}{6!}+\dots = \sum_{n=0}^{\infty}(-1)^n\frac{x(2x)^{2n}}{(2n)!}$$

$$\boxed{f(x)=x\cos 2x = x-\frac{2^2x^3}{2!}+\frac{2^4x^5}{4!}-\frac{2^6x^7}{6!}+\dots = \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}x^{2n+1}}{(2n)!}}$$

36.// First, convert degrees to radians: $3° = \pi/60$ radians

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\sin(\pi/60) = \pi/60 - \frac{(\pi/60)^3}{3!} + \frac{(\pi/60)^5}{5!} - \cdots$$

We see $\left|\frac{(\pi/60)^5}{5!}\right| < 10^{-5}$ so by the Alternating Series Estimation Theorem, we truncate before this term.

$$\boxed{\sin 3° = \sin \pi/60 \approx \pi/60 - (\pi/60)^3/3! \approx 0.05234}$$

FYI: The error bound 10^{-5} may induce rounding error, but for the purposes of this exercise (and others like it) we will use the convention: "to N decimal places" requires error less than or equal to 10^{-N}.

40.// $\int e^{x^3}dx$

We know $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{x^3} = 1 + x^3 + \frac{(x^3)^2}{2!} + \frac{(x^3)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!}$$

$$\boxed{\int e^{x^3}dx = \underline{\underline{C}} + x + \tfrac{1}{4}x^4 + \tfrac{1}{7}\cdot\frac{x^7}{2!} + \tfrac{1}{10}\cdot\frac{x^{10}}{3!} + \cdots = C + \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)n!}}$$

54.// $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$

Can be written: $\sum_{n=0}^{\infty} (-1)^n \frac{(\pi/6)^{2n}}{(2n)!} = \cos \pi/6 = \boxed{\sqrt{3}/2}$

Optional:

(4.)// $f(x) = \sin 2x$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin 2x$	0
1	$2\cos 2x$	2
2	$-2^2 \sin 2x$	0
3	$-2^3 \cos 2x$	-2^3
4	$2^4 \sin 2x$	0
...	...	...

Maclaurin:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)x^3}{3!} + \cdots$$

$$\boxed{\sin 2x = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \cdots}$$

or alternatively, notice $\sin x = x - x^3/3! + x^5/5! - \cdots$

so $\sin(2x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots$

Use ratio test for radius of convergence:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^2|x|^2}{(2n+3)(2n+2)} = 0 < 1 \text{ for all } x$$

so $\boxed{R = \infty}$

(12.)// $f(x) = \ln x$ @ $x = 2$

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	x^{-1}	$1/2$
2	$-x^{-2}$	$-1/4$
3	$2x^{-3}$	$2/8$
4	$-3\cdot 2x^{-4}$	$-3\cdot 2/16$
...	...	...

$$\boxed{\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{n\cdot 2^n}}$$

(45) $\lim_{x\to 0} \dfrac{x - \tan^{-1}x}{x^3}$

notice: $\tan^{-1}x$ ($=\arctan x$ NOT $\frac{1}{\tan x}$) $= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

$-\tan^{-1}x = -x + x^3/3 - x^5/5 + x^7/7 - \cdots$

$x - \tan^{-1}x = x^3/3 - x^5/5 + x^7/7 - \cdots$

$\dfrac{x-\tan^{-1}x}{x^3} = \frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} - \cdots$

So $\lim_{x\to 0} \dfrac{x-\tan^{-1}x}{x^3} = \frac{1}{3} - \frac{(0)^2}{5} + \frac{(0)^4}{7} - \cdots = \boxed{\frac{1}{3}}$

(56) $\sum_{n=0}^{\infty} \dfrac{3^n}{5^n n!}$

$\sum_{n=0}^{\infty} \dfrac{3^n}{5^n n!} = \sum_{n=0}^{\infty} \dfrac{(3/5)^n}{n!} = \boxed{e^{3/5}}$