11.3 6, (8), 12, (21), 18, 22, 38

6. $f(x) = e^{-x}$ continuous, positive, decreasing on $[1, \infty)$

$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]_1^b$$

$$= \lim_{b\to\infty} \left(-e^{-b} + \frac{1}{e}\right) = \frac{1}{e} \quad \text{so } \sum_{n=1}^{\infty} e^{-n} \ \underline{\text{converges}}$$

12. $$1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

p-series $\therefore p = 3/2 > 1$ so converges.

18. $f(x) = \dfrac{1}{2x^2+3x+1} = \dfrac{1}{(2x+1)(x+1)}$ positive, continuous, decreasing on $[1, \infty)$

$$\int_1^\infty f(x)\,dx = \lim_{b\to\infty} \int_1^b \left(\frac{2}{2x+1} - \frac{1}{x+1}\right) dx$$

$$= \lim_{b\to\infty} \left[\ln(2x+1) - \ln(x+1)\right]_1^b = \lim_{b\to\infty} \left[\ln\left(\frac{2x+1}{x+1}\right)\right]_1^b$$

$$= \lim_{b\to\infty} \left(\ln \frac{2b+1}{b+1} - \ln 3/2\right) = \ln 2 - \ln 3/2$$

$$= \ln 4/3 \quad \underline{\text{converges}}$$

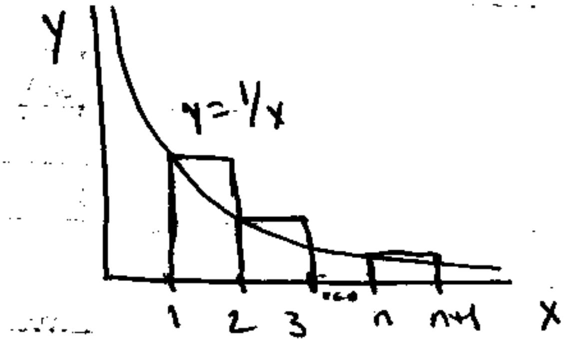
22. $f(x) = \dfrac{\ln x}{x^2}$ cts, pos, decreasing for $x \geq 2$

$f'(x) = \dfrac{1 - 2\ln x}{x^3}$

$$\int_2^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t = 1 \text{ by L'Hospital}$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2} = \sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ converges.

38.
a.



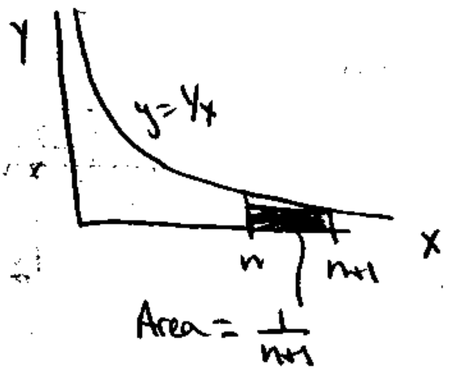
Sum of areas of rectangles: $1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}$

$\int_1^{n+1}\frac{dx}{x}$ less than this sum since rectangles extend above curve $y=1/x$ so

$$\int_1^{n+1}\frac{dx}{x}=\ln(n+1)<1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} \text{ ; since}$$

$$\ln n<\ln(n+1) \therefore 0<1+\frac{1}{2}+\dots+\frac{1}{n}-\ln n=t_n$$

b.



Area under $y=1/x$ between $x=n$ and $x=n+1$ is:

$$\int_n^{n+1}\frac{dx}{x}=\ln(n+1)-\ln n$$

which is greater than the area of the inscribed rectangle

So $t_n-t_{n+1}=[\ln(n+1)-\ln n]-\frac{1}{n+1}>0$

so $t_n>t_{n+1}$ so $\{t_n\}$ decreasing seq

c. We've shown $\{t_n\}$ decreasing sequence and $t_n > 0 \ \forall n$. Thus $0 < t_n \leq t_1 = 1$ so $\{t_n\}$ is a bounded monotonic seq $\therefore$ converges.

Optional Problems

8. $\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{4n-1}$

$f(x) = \frac{1}{4x-1}$ pos, cts, decreasing on $[1, \infty)$

$$\int_1^\infty \frac{dx}{4x-1} = \lim_{b\to\infty} \int_1^b \frac{dx}{4x-1} = \lim_{b\to\infty} \left[\tfrac{1}{4}\ln(4x-1)\right]_1^b$$

$$= \lim_{b\to\infty} \left[\tfrac{1}{4}\ln(4b-1) - \tfrac{1}{4}\ln 3\right] = \infty \quad \text{diverges}$$

so series diverges

21. $f(x) = \frac{\arctan x}{1+x^2}$ cts, pos on $[1, \infty)$

$f'(x) = \frac{1 - 2x\arctan x}{(1+x^2)^2} < 0$ for $x > 1$ since

$2x\arctan x \geq \frac{\pi}{2} > 1$ for $x \geq 1$

$$\int_1^\infty \frac{\arctan x}{1+x^2}\,dx = \lim_{t\to\infty} \left[\tfrac{1}{2}\arctan^2 x\right]_1^t = \frac{(\pi/2)^2}{2} - \frac{(\pi/4)^2}{2}$$

$$= \frac{3\pi^2}{32}$$

so series converges