Solutions to §17.4 : 2, 4, 8, 10

2. Let $y(x)=\sum_{n=0}^{\infty} c_n x^n$. Then $\sum_{n=0}^{\infty} nc_n x^{n-1} - x\sum_{n=0}^{\infty} c_n x^n = 0$ or $\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$. Replacing n by $n+1$ in the first sum and n by $n-1$ in the second gives $\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n = 0$ or $c_1 + \sum_{n=1}^{\infty}(n+1)c_{n+1}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n = 0$. Thus $c_1 + \sum_{n=1}^{\infty}\left[(n+1)c_{n+1} - c_{n-1}\right]x^n = 0$. Equating coefficients gives $c_1 = 0$ and $(n+1)c_{n+1} - c_{n-1} = 0$. Thus the recursion relation is $c_{n+1} = \dfrac{c_{n-1}}{n+1}$, $n = 1, 2, \ldots$. But $c_1 = 0$, so $c_3 = 0$ and $c_5 = 0$ and in general $c_{2n+1} = 0$. Also $c_2 = \dfrac{c_0}{2}$, $c_4 = \dfrac{c_2}{4} = \dfrac{c_0}{4\cdot 2} = \dfrac{c_0}{2^2\cdot 2!}$, $c_6 = \dfrac{c_4}{6} = \dfrac{c_0}{6\cdot4\cdot2} = \dfrac{c_0}{2^3\cdot 3!}$ and in general $c_{2n} = \dfrac{c_0}{2^n\cdot n!}$. Thus the solution is $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n = \sum\limits_{n=0}^{\infty} c_{2n}x^{2n} = c_0\sum\limits_{n=0}^{\infty}\dfrac{(x^2/2)^n}{n!} = c_0 e^{x^2/2}$.

4. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n$. Then the differential equation becomes $(x-3)\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0 \Rightarrow$
$\sum_{n=0}^{\infty}(n+1)c_{n+1}x^{n+1} - 3\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0 \Rightarrow$
$\sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} 3(n+1)c_{n+1}x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0 \Rightarrow$
$\sum_{n=0}^{\infty}\left[(n+2)c_n - 3(n+1)c_{n+1}\right]x^n = 0$ (since $\sum_{n=1}^{\infty} nc_n x^n = \sum_{n=0}^{\infty} nc_n x^n$). Equating coefficients gives $(n+2)c_n - 3(n+1)c_{n+1} = 0$, thus the recursion relation is $c_{n+1} = \dfrac{(n+2)c_n}{3(n+1)}$, $n = 0, 1, 2, \ldots$. Then $c_1 = \dfrac{2c_0}{3}$, $c_2 = \dfrac{3c_1}{3(2)} = \dfrac{3c_0}{3^2}$, $c_3 = \dfrac{4c_2}{3(3)} = \dfrac{4c_0}{3^3}$, $c_4 = \dfrac{5c_3}{3(4)} = \dfrac{5c_0}{3^4}$, and in general, $c_n = \dfrac{(n+1)c_0}{3^n}$. Thus the solution is $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n = c_0\sum\limits_{n=0}^{\infty}\dfrac{n+1}{3^n}x^n$. [Note that $c_0\sum\limits_{n=0}^{\infty}\dfrac{n+1}{3^n}x^n = \dfrac{9c_0}{(3-x)^2}$ for $|x| < 3$.]

8. Assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$, $y''(x) = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$ and $-xy(x) = -\sum_{n=0}^{\infty} c_n x^{n+1} = -\sum_{n=1}^{\infty} c_{n-1}x^n$. The differential equation becomes $\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n = 0$ or $c_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)c_{n+2} - c_{n-1}\right]x^n = 0$. Equating coefficients gives $c_2 = 0$ and $c_{n+2} = \dfrac{c_{n-1}}{(n+2)(n+1)}$ for $n = 1, 2, \ldots$. Since $c_2 = 0$, $c_{3n+2} = 0$ for $n = 0, 1, 2, \ldots$. Given c_0 and c_1, $c_3 = \dfrac{c_0}{3\cdot2}$, $c_6 = \dfrac{c_3}{6\cdot5} = \dfrac{c_0}{6\cdot5\cdot3\cdot2}, \ldots,$
$c_{3n} = \dfrac{c_0}{3n(3n-1)(3n-3)(3n-4)\cdot\cdots\cdot6\cdot5\cdot3\cdot2}$; also $c_4 = \dfrac{c_1}{4\cdot3}$, $c_7 = \dfrac{c_4}{7} = \dfrac{c_1}{7\cdot6\cdot4\cdot3}$,
$\ldots, c_{3n+1} = \dfrac{c_1}{(3n+1)3n(3n-2)(3n-3)\ldots7\cdot6\cdot4\cdot3}$. The solution is
$$y(x) = c_0\sum_{n=0}^{\infty}\frac{(3n-2)(3n-5)\cdot\cdots\cdot7\cdot4}{(3n)!}x^{3n} + c_1\sum_{n=0}^{\infty}\frac{(3n-1)(3n-4)\cdot\cdots\cdot8\cdot5\cdot2}{(3n+1)!}x^{3n+1}.$$

10. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$, $y''(x) = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=-2}^{\infty}(n+4)(n+3)c_{n+4}x^{n+2}$ $= 2c_2 + 6c_3x + \sum_{n=0}^{\infty}(n+4)(n+3)c_{n+4}x^{n+2}$. Thus the differential equation becomes $2c_2 + 6c_3x + \sum_{n=0}^{\infty}\left[(n+4)(n+3)c_{n+4} + c_n\right]x^{n+2} = 0$. So $c_2 = c_3 = 0$ and the recursion relation is $c_{n+4} = -\dfrac{c_n}{(n+4)(n+3)}$, $n = 0, 1, 2, \ldots$. But $c_1 = y'(0) = 0 = c_2 = c_3$ and by the recursion relation $c_{4n+1} = c_{4n+2} = c_{4n+3} = 0$ for $n = 0, 1, 2, \ldots$. Also $c_0 = y(0) = 1$, so $c_4 = -\dfrac{1}{4\cdot3}$, $c_8 = -\dfrac{c_4}{8\cdot7} = \dfrac{(-1)^2}{8\cdot7\cdot4\cdot3}, \ldots, c_{4n} = \dfrac{(-1)^n}{4n(4n-1)(4n-4)(4n-5)\ldots4\cdot3}$. Thus the solution to the initial-value problem is $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n = 1 + \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{4n}}{4n(4n-1)\ldots4\cdot3}$.