SOLNS: 9.6 LINEAR EQUATIONS

Optional

① $y' + e^x y = x^2 y^2$ is not linear — cannot be put into form $y' + P(x)y = Q(x)$. (The y^2 term ruins it)

② $y + \sin x = x^3 y' \rightarrow x^3 y' - y = \sin x \rightarrow y' + \left(-\frac{1}{x^3}\right) y = \frac{\sin x}{x^3}$

Can be put into standard linear form, so linear

③ $xy' + \ln x - x^2 y = 0 \rightarrow xy' - x^2 y = -\ln x \rightarrow y' + (-x)y = -\frac{\ln x}{x}$

Can be put into standard linear form, so linear

④ $yy' = \sin x$ is not linear since it cannot be put into standard linear form (y cannot multiply y' — ruins linearity)

⑥ $y' = x + 5y \rightarrow y' + (-5)y = x$

So $I(x) = e^{\int -5\,dx} = e^{-5x}$

Multiply differential equation by $I(x)$:

$$e^{-5x} y' + (-5)e^{-5x} y = xe^{-5x}$$

$$\frac{d}{dx}(e^{-5x} y) = xe^{-5x}$$

Integrate both sides ($\int xe^{-5x}dx$ by parts: $u = x$, $dv = e^{-5x}dx$, $du = dx$, $v = -\frac{1}{5}e^{-5x}$

$\int xe^{-5x}dx = x(-\frac{1}{5}e^{-5x}) - \int -\frac{1}{5}e^{-5x}dx = -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + C$)

$$e^{-5x} y = \int xe^{-5x}$$

$$e^{-5x} y = -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + C$$

$$\boxed{y = -\frac{1}{5}x - \frac{1}{25} + Ce^{5x}}$$

⑧ $y' + \frac{2}{x}y = \frac{1}{x}e^{x^2}$ $(x \neq 0$

So $I(x) = e^{\int 2x^{-1}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

Multiplying both sides by $I(x)$:

$$x^2 y' + 2xy = xe^{x^2}$$

$$\frac{d}{dx}(x^2 y) = xe^{x^2}$$

$$x^2 y = \int xe^{x^2}$$

$$x^2 y = \frac{1}{2}e^{x^2} + C$$

$$y = \frac{e^{x^2} + C_1}{2x^2}$$

⑬ $(1+t)\frac{du}{dt} + u = 1+t \rightarrow \frac{d}{dt}((1+t)u) = 1+t$

Integrating wrt t:

$$(1+t)u = \int (1+t)dt$$

$$(1+t)u = t + \frac{1}{2}t^2 + C$$

$$\boxed{u = \frac{t + \frac{1}{2}t^2 + C}{1+t}}$$

18) $y' + \frac{2x}{1+x^2}y = \frac{3\sqrt{x}}{1+x^2}$

$I(x) = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln|1+x^2|} = 1+x^2$

Multiplying both sides by $I(x)$,

$$(1+x^2)y' + 2xy = 3\sqrt{x}$$

$$\frac{d}{dx}\left((1+x^2)y\right) = 3\sqrt{x}$$

$$(1+x^2)y = \int 3x^{1/2}dx$$

$$(1+x^2)y = \cancel{3}\left(\frac{2}{\cancel{3}}\right)x^{3/2} + C$$

$$y = \frac{2x^{3/2}+C}{1+x^2}$$

$y(0) = \frac{2(0)+C}{1+0^2} = 2 \rightarrow C=2$, so $\boxed{y = \frac{2x^{3/2}+2}{1+x^2}}$

24) $xy' + y = -xy^2 \rightarrow y' + \frac{1}{x}y = -y^2$

Try substitution $u = y^{1-2} = y^{-1}$

Then $\frac{du}{dx} = -y^{-2}\frac{dy}{dx} \rightsquigarrow -y^2u' = y' \rightarrow -\frac{u'}{u^2} = y'$

So $\left(-\frac{u'}{u^2}\right) + \frac{1}{x}\left(\frac{1}{u}\right) = -\left(\frac{1}{u}\right)^2$

Multiply by $-u^2$: $u' - \frac{1}{x}u = 1$, as shown in #23 ($n=2$, $P(x)=1/x$, $Q(x)=-1$)

Then $I(x) = e^{\int -1/x\,dx} = e^{-\ln x} = e^{\ln(1/x)} = 1/x$

Multiplying by $I(x)$,

$$\frac{1}{x}u' - \frac{1}{x^2}u = \frac{1}{x}$$

$$\frac{d}{dx}\left(\frac{1}{x}u\right) = \frac{1}{x}$$

$$\frac{1}{x}u = \int \frac{1}{x}dx = \ln|x| + C$$

$$u = x(\ln|x| + C)$$

And $y = u^{-1}$, so $\boxed{y = \frac{1}{x(\ln|x|+C)}}$

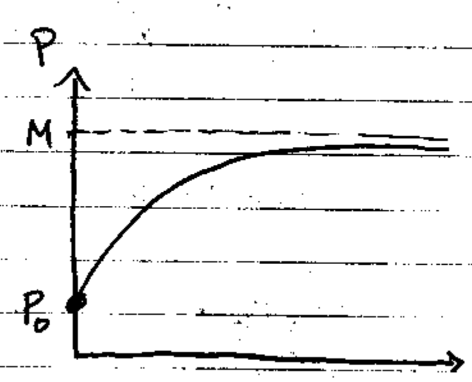
optional 31) $\frac{dP}{dt} + kP = kM$, so $I(t) = e^{\int k\,dt} = e^{kt}$

$$e^{kt}P' + ke^{kt}P = kMe^{kt}$$

$$\int (e^{kt}P)'_{dt} = \int kMe^{kt}dt$$

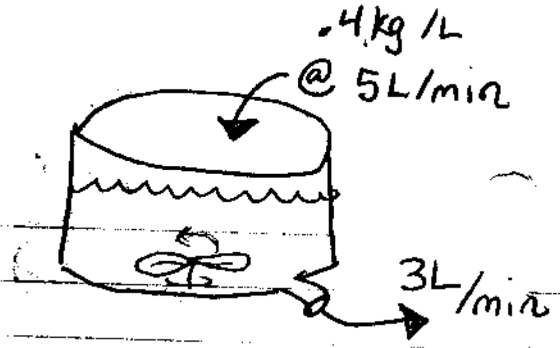
$$e^{kt}P = Me^{kt} + C$$

$$P = M + Ce^{-kt}$$



We assume $0 \le P(0) \le M$, so that you gain knowledge, instead of lose, so $-M \le C \le 0$

(33) Let $y(t)$ = amt of salt



$\frac{dy}{dt}$ = rate in − rate out

rate in $= (.4\frac{kg}{L})(5\frac{L}{min}) = 2\frac{kg}{min}$

rate out $= \left(\frac{y(t)}{\text{amt } \cancel{\text{liquid}}}\right)\left(\frac{3L}{min}\right)$

How much liquid is in tank at time t?
Initially 100 L, and 2L/min added, so am't liquid = 100L + 2t L/min

So $\boxed{\frac{dy}{dt} = 2 - \frac{3y}{100+2t}}$

$$y' + \frac{3}{100+2t}y = 2$$

$$I(t) = e^{\int\left(\frac{3}{100+2t}\right)dt} = e^{\frac{3}{2}\ln|100+2t|} = e^{\ln(100+2t)^{3/2}} = (100+2t)^{3/2}$$

So $(100+2t)^{3/2}y' + (100+2t)^{1/2}\,3y = 2(100+2t)^{3/2}$

$$[(100+2t)^{3/2}y]' = 2(100+2t)^{3/2}$$

$$(100+2t)^{3/2}y = \int 2(100+2t)^{3/2}dt$$

$$(100+2t)^{3/2}y = \frac{2}{5}(100+2t)^{5/2} + C$$

$$y = \frac{2}{5}(100+2t) + C(100+2t)^{-3/2}$$

Now $y(0) = 0 = \frac{2}{5}(100) + C(100)^{-3/2}$

$$-40 = C/1000 \rightarrow C = -40000$$

So $\boxed{y = \frac{2}{5}(100+2t) - 40000(100+2t)^{-3/2}}$ in kg

Concentration of salt is $\boxed{\frac{y}{100+2t} = \frac{2}{5} - 40000(100+2t)^{-5/2}}$

So at $t=20$, concentration is $\frac{2}{5} - 40000(100+40)^{-5/2} \approx .2275\ \frac{kg}{L}$

And total am't salt is $y = \frac{2}{5}(140) - 40000(140)^{-3/2} \approx 31.85\text{kg}$

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optional (35) $\frac{dv}{dt} + \frac{c}{m}v = g$, so $I(t) = e^{\int\frac{c}{m}dt} = e^{(c/m)t}$

$$e^{(c/m)t}v' + \frac{c}{m}e^{(c/m)t}v = ge^{(c/m)t}$$

$$\int(e^{(c/m)t}v)'dt = \int ge^{c/m\,t}dt$$

$$e^{(c/m)t}v = \frac{mg}{c}e^{(c/m)t} + k \rightarrow \boxed{v = \frac{mg}{c} + Ke^{-(c/m)t}}$$

Object is dropped from rest: $v(0) = 0 = \frac{mg}{c} + K \rightarrow K = -mg/c$

So $\boxed{v(t) = \frac{mg}{c}\left(1 - e^{-(c/m)t}\right)}$

(b) $\boxed{\lim_{t\to\infty} v(t) = mg/c}$

(c) $s(t) = \int v(t)dt = \boxed{\frac{mg}{c}\left(t + \frac{m}{c}e^{-(c/m)t}\right) + k_s}$

$s(0) = \frac{m^2g}{c^2} + k_s = 0$

$\boxed{k_s = -\frac{m^2g}{c^2}}$

So $s(t) = \frac{mg}{c}\left(t + \frac{m}{c}e^{-(c/m)t}\right) - m^2g/c^2$