**Problem:** Evaluate $\int \frac{1}{x^2+a}\,dx$.

**Solution:**

**The Case $a < 0$:**

If $a < 0$, then the discriminant of the polynomial is $-4a > 0$. Therefore the denominator factors into two distinct linear factors (which can be found by means of the quadratic formula). Hence we obtain

$$\int \frac{1}{x^2+a}\,dx = \int \frac{1}{\left(x-\sqrt{|a|}\right)\left(x+\sqrt{|a|}\right)}\,dx.$$

Using the method of partial fractions, we get

$$\frac{1}{\left(x-\sqrt{|a|}\right)\left(x+\sqrt{|a|}\right)} = \frac{A}{x-\sqrt{|a|}} + \frac{B}{x+\sqrt{|a|}},$$

which leads to

$$1 = A\left(x+\sqrt{|a|}\right) + B\left(x-\sqrt{|a|}\right).$$

By plugging in the values $x = \sqrt{|a|}$ and $x = -\sqrt{|a|}$, we see that $A = \frac{1}{2\sqrt{|a|}}$ and $B = -\frac{1}{2\sqrt{|a|}}$. Therefore,

$$\begin{aligned}
\int \frac{1}{x^2+x+a}\,dx &= \int \left(\frac{\frac{1}{2\sqrt{|a|}}}{x-\sqrt{|a|}} - \frac{\frac{1}{2\sqrt{|a|}}}{x+\sqrt{|a|}}\right)dx \\
&= \frac{1}{2\sqrt{|a|}}\ln\left|x-\sqrt{|a|}\right| - \frac{1}{2\sqrt{|a|}}\ln\left|x+\sqrt{|a|}\right| + C \\
&= \frac{1}{2\sqrt{|a|}}\ln\left|\frac{x-\sqrt{|a|}}{x+\sqrt{|a|}}\right| + C.
\end{aligned}$$

**The Case $a > 0$:**

If $a > 0$, then the discriminant of the polynomial is $-4a < 0$. Therefore the denominator is an irreducible quadratic, and so we must integrate using arctangents.

$$\int \frac{1}{x^2+a}\,dx = \frac{1}{\sqrt{a}}\tan^{-1}\frac{x}{\sqrt{a}} + C$$

.

**The Case $a = 0$:**

If $a = 0$, then the discriminant of the polynomial is 0, and so we have a repeated root of $a = 0$. Thus the integral is simply

$$\int \frac{1}{x^2+a}\,dx = \int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

**Summary:**

We have

$$\int \frac{1}{x^2+a}\,dx = \begin{cases} \frac{1}{2\sqrt{|a|}} \ln\left|\frac{x-\sqrt{|a|}}{x+\sqrt{|a|}}\right| + C & \text{if } a<0; \\ \frac{1}{\sqrt{a}} \tan^{-1} \frac{x}{\sqrt{a}} + C & \text{if } a>0; \\ -\frac{1}{x} + C & \text{if } a=0. \end{cases}$$