Problem: Evaluate $\int \frac{1}{x^2+x+a}\,dx$.

Solution:

The Case $a < \frac{1}{4}$:

If $a < \frac{1}{4}$, then the discriminant of the polynomial is $1 - 4a > 0$. Therefore the denominator factors into two distinct linear factors (which can be found by means of the quadratic formula). Hence we obtain

$$\int \frac{1}{x^2+x+a}\,dx = \int \frac{1}{\left(x - \frac{-1+\sqrt{1-4a}}{2}\right)\left(x - \frac{-1-\sqrt{1-4a}}{2}\right)}\,dx.$$

Using the method of partial fractions, we get

$$\frac{1}{\left(x - \frac{-1+\sqrt{1-4a}}{2}\right)\left(x - \frac{-1-\sqrt{1-4a}}{2}\right)} = \frac{A}{x - \frac{-1+\sqrt{1-4a}}{2}} + \frac{B}{x - \frac{-1-\sqrt{1-4a}}{2}},$$

which leads to

$$1 = A\left(x - \frac{-1-\sqrt{1-4a}}{2}\right) + B\left(x - \frac{-1+\sqrt{1-4a}}{2}\right).$$

By plugging in the values $x = \frac{-1+\sqrt{1-4a}}{2}$ and $x = \frac{-1-\sqrt{1-4a}}{2}$, we see that $A = \frac{1}{\sqrt{1-4a}}$ and $B = -\frac{1}{\sqrt{1-4a}}$. Therefore,

$$\begin{aligned}
\int \frac{1}{x^2+x+a}\,dx &= \int \left(\frac{\frac{1}{\sqrt{1-4a}}}{x - \frac{-1+\sqrt{1-4a}}{2}} - \frac{\frac{1}{\sqrt{1-4a}}}{x - \frac{-1-\sqrt{1-4a}}{2}}\right) dx \\
&= \frac{1}{\sqrt{1-4a}}\left(\ln\left|x - \frac{-1+\sqrt{1-4a}}{2}\right| + \ln\left|x - \frac{-1-\sqrt{1-4a}}{2}\right|\right) + C \\
&= \frac{1}{\sqrt{1-4a}} \ln\left|\frac{x - \frac{-1+\sqrt{1-4a}}{2}}{x - \frac{-1-\sqrt{1-4a}}{2}}\right| + C.
\end{aligned}$$

The Case $a > \frac{1}{4}$:

If $a > \frac{1}{4}$, then the discriminant of the polynomial is $1 - 4a < 0$. Therefore the denominator is an irreducible quadratic, and so we must use the method of completing the squre. We have

$$x^2 + x + a = \left(x + \frac{1}{2}\right)^2 + a - \frac{1}{4}.$$

Thus we want to change variables to $u = x + \frac{1}{2}$ to obtain

$$\int \frac{1}{x^2+x+a}\,dx = \int \frac{1}{u^2+c^2}\,du,$$

where

$$c = \sqrt{a - \frac{1}{4}} = \frac{\sqrt{4a-1}}{2}.$$

This integral can then be evaluated in terms of the arctangent,

$$\begin{aligned}\int \frac{1}{x^2+x+a}\,dx &= \frac{1}{c}\tan^{-1}\frac{u}{c}+C \\ &= \frac{2}{\sqrt{4a-1}}\tan^{-1}\left(\frac{2x+1}{\sqrt{4a-1}}\right).\end{aligned}$$

The Case $a = \frac{1}{4}$:

If $a = \frac{1}{4}$, then the discriminant of the polynomial is 0, and so we have a repeated root of $a = \frac{1}{2}$. Thus the integral is simply

$$\int \frac{1}{x^2+x+a}\,dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2}\,dx = -\frac{1}{x+\frac{1}{2}}+C.$$

Summary:

We have

$$\int \frac{1}{x^2+x+a}\,dx = \begin{cases} \frac{1}{\sqrt{1-4a}}\ln\left|\frac{x-\frac{-1+\sqrt{1-4a}}{2}}{x-\frac{-1-\sqrt{1-4a}}{2}}\right|+C & \text{if } a<1/4; \\ \frac{2}{\sqrt{4a-1}}\tan^{-1}\left(\frac{2x+1}{\sqrt{4a-1}}\right) & \text{if } a>1/4; \\ \int \frac{1}{\left(x+\frac{1}{2}\right)^2}\,dx = -\frac{1}{x+\frac{1}{2}}+C & \text{if } a=1/4. \end{cases}$$