Harvard University	Your Name: ____________________
Math 1b	Your TF's Name: __________________
Spring, 2000	March 14, 2000

First Exam

- Do not open this exam booklet until you are directed to do so.
- You have 120 minutes to earn 100 points.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem (and explain clearly that you are doing so). Do not put part of the answer to one problem on the back of the sheet for another problem.
- Do not spend too much time on any problem. Read through them all first and attack them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given (unless indicated otherwise). You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.
- No calculators are allowed on this exam.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	15	
6	10	
7	15	

Some Useful Formulas

- (Error Bound for Taylor Polynomials) When we approximate $f(x)$ by the n-th degree Taylor polynomial about x_0, the error is at most

$$\frac{M|x - x_0|^{n+1}}{(n+1)!},$$

where M is an upper bound for $|f^{(n+1)}|$ between x_0 and x.

- (L'Hospital's Rule) If $f(x_0) = g(x_0) = 0$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)},$$

if f and g are differentiable and the limit on the right exists.

1. (15 points)

1a. (8 points) What is the radius of convergence of

$$\sum_{m=1}^{\infty} \frac{x^m}{m^2}?$$

For which values of x does this series converge?

1b. (7 points) Find a Taylor series whose interval of convergence is $[2, 8]$. (In other words, it must converge for $2 \le x \le 8$ and no other x.)

2. (15 points)

Recall from class that

$$\sum_{j=1}^{n} x^j = \frac{x - x^{n+1}}{1 - x}.$$

2a. (5 points) Using this formula, compute

$$\sum_{j=1}^{n} j x^{j-1}.$$

2b. (5 points) Using your result in part a, take a limit in order to compute the value of the sum

$$\sum_{j=1}^{\infty} j \left(\frac{1}{2}\right)^{j-1}.$$

2c. (5 points) Using your result from part a, take the limit as $x \to 1$ and use L'Hospital's rule to find a formula for

$$\sum_{j=1}^{n} j.$$

3. (15 points) For each of these series, indicate whether it converges or diverges. No justification need be given, and no partial credit will be awarded. Remember that each of the fifteen series is worth only one point, so do not spend too much time on any one of them.

$$\sum_{a=36}^{\infty} 17\pi$$

$$\sum_{b=9}^{\infty} \frac{4^b}{3^{2b}}$$

$$\sum_{c=32}^{\infty} \frac{1}{(c!)^{3/8}}$$

$$\sum_{d=13}^{\infty} (\log d)^{-d}$$

$$\sum_{f=12}^{\infty} \frac{1}{\log f}$$

$$\sum_{g=6}^{\infty} \frac{g^4+1}{g^6-1}$$

$$\sum_{h=3}^{\infty} \frac{(-1)^h}{h}$$

$$\sum_{i=4}^{\infty} \frac{(-1)^{2i+1}}{\sqrt{i}}$$

$$\sum_{j=88}^{\infty} 173589^{-5/j^2}$$

$$\sum_{k=5}^{\infty} \frac{(k-1)!}{k!}$$

4. (15 points)

4a. (5 points) Compute the 12-th degree Taylor polynomial of $e^{(x^3)}$ about $x = 0$. (Hint: make use of the Taylor series for e^x.)

4b. (10 points) Compute the 4-th degree Taylor polynomial of $\log(\sin x)$ about $x = \pi/2$.

5. (15 points)

5a. (10 points) Compute $\sin 0.1$ to four decimal places after the decimal point.

5b. (5 points) Suppose that the function f satisfies

$$f^{(6)}(x) = \sin(e^{2x}).$$

If one tries to approximate $f(2)$ using the 5-th degree Taylor polynomial for $f(x)$ about $x = 0$, how much error could there possibly be? Give a bound using the general error term bound for Taylor polynomials.

6. (10 points)
6a. (5 points) Give the first three **non-zero** terms in the MacLaurin series for the function

$$f(x) = \frac{1}{\sqrt[3]{1-x^2}}.$$

6b. (5 points) Suppose that $g(x)$ is a function that has $g(0) = 4$, $g'(0) = -3$, and $g''(0) = 2$. What are the first three **non-zero** terms in the MacLaurin series for the function $h(x) = g(x)f(x)$, where $f(x)$ is the function from part a. (Hint: You might want to try writing out the first few terms in the MacLaurin series for $g(x)$ first.)

7. (15 points)

7a. (5 points) Does

$$\sum_{k=1}^{\infty} \frac{2\cos(k^2)+1}{k^2}$$

converge? Why or why not? (Be sure to give justification for your answer.)

7b. (5 points) Compute

$$\sum_{\ell=0}^{\infty} \frac{2^{\ell+2}3^{2\ell}}{5^{\ell-1}7^{\ell+1}}.$$

7c. (5 points) Compute

$$\sum_{n=1}^{\infty} \left(\frac{\cos n}{13e^n} - \frac{\cos(n-1)}{13e^{n-1}} \right).$$