Harvard University
Math 1b. Second Semester Calculus
Exam One

Name Answer Key (A Popa)

March 17, 1999

Exam One

- Do not open this test booklet until you are directed to do so.
- You have 2 hours. The test is out of 100 points.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please don't put part of the answer to one problem on the back of the sheet for another problem.
- Don't spend too much time on any one problem. Read them all through first and work on them in the order that allows you to make the most progress.
- Be sure to show all of your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Try to be neat. By the same token, be sure to justify your solutions (unless you are explicitly told otherwise), so that the graders can follow your reasoning.
- Good luck!

Problem	Points	Grade
1	10	
2	17	
3	15	
4	15	
5	14	
6	14	
7	15	
Total	100	

Please circle your section:

MWF 10:00	MWF 11:00	MWF 12:00	TTh 10:00	TTh 11:30
Brian Conrad	Grisha Mikhalkin	Cathy O'Neil	Andy Engelward	Andy Engelward

1. (10 pts) For each of the following infinite series, determine if it converges or diverges. You need to justify your answer to receive full credit. You don't need to specify whether convergence is conditional or absolute, just whether the series converges or diverges.

a. (2 pts) $\sum_{k=1}^{\infty} e^{\frac{1}{k}}$

$\lim_{K\to\infty} e^{\frac{1}{K}} = e^{\lim_{K\to\infty}\frac{1}{K}} = e^0 = 1$, so the series diverges by the divergence test.

b. (2 pts) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$

This is an alternating series ($\frac{1}{\sqrt{1}} > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} \cdots$ and $\lim_{K\to\infty} \frac{1}{\sqrt{K}} = 0$)

so the series CONVERGES

c. (3 pts) $\sum_{k=1}^{\infty} \frac{|\sin(k)|}{k^4}$

THE SERIES CONVERGES BY COMPARISON WITH THE P-SERIES $\sum_{K=1}^{\infty} \frac{1}{K^4}$: $|\sin K| \leq 1$, so $\frac{|\sin K|}{K^4} \leq \frac{1}{K^4}$

d. (3 pts) $\sum_{k=0}^{\infty} \frac{1}{\sqrt{k^2+1}}$

THE SERIES DIVERGES BY LIMIT COMPARISON WITH THE HARMONIC SERIES $\sum_{K=1}^{\infty} \frac{1}{K}$: $\lim_{K\to\infty} \frac{\frac{1}{K}}{\frac{1}{\sqrt{K^2+1}}} = \lim_{K\to\infty} \frac{\sqrt{K^2+1}}{K} = \lim_{K\to\infty} \sqrt{1+\frac{1}{K^2}} =$

$= 1 < \infty$
$\neq 0$

2. (17 pts)

a. (3 pts each) Evaluate each of the following infinite series.

i. $\sum_{k=0}^{\infty}(-1)^k 6^{-2k-2}$

WRITE IT AS: $6^{-2}\sum_{K=0}^{\infty}\left(-\frac{1}{6^2}\right)^K = 6^{-2}\frac{1}{1+\frac{1}{6^2}} = \frac{1}{6^2+1} = \boxed{\frac{1}{37}}$

ii. $\sum_{k=0}^{\infty}\frac{2^k\, 5^{k+1}}{3^{k+2}\, 7^k}$

WRITE AS: $\sum_{K=0}^{\infty}\frac{5}{9}\cdot\frac{2^K\cdot 5^K}{3^K\cdot 7^K} = \frac{5}{9}\sum_{K=0}^{\infty}\left(\frac{10}{21}\right)^K = \frac{5}{9}\cdot\frac{1}{1-\frac{10}{21}} = \frac{5}{9}\cdot\frac{21}{11} = \boxed{\frac{35}{33}}$

iii. $\sum_{k=3}^{\infty}\frac{1-2^k}{3^k} = S$

WRITE THE SERIES AS A DIFFERENCE OF GEOMETRIC SERIES

$$S = \sum_{K=3}^{\infty}\frac{1}{3^K} - \sum_{K=3}^{\infty}\left(\frac{2}{3}\right)^K = \frac{1}{3^3}\sum_{K=3}^{\infty}\frac{1}{3^{K-3}} - \frac{2^3}{3^3}\sum_{K=3}^{\infty}\left(\frac{2}{3}\right)^{K-3}$$

$$= \frac{1}{3^3}\sum_{K=0}^{\infty}\frac{1}{3^K} - \frac{2^3}{3^3}\sum_{K=0}^{\infty}\left(\frac{2}{3}\right)^K$$

$$= \frac{1}{27}\cdot\frac{1}{1-\frac{1}{3}} - \frac{8}{27}\cdot\frac{1}{1-\frac{2}{3}} = \frac{1}{18} - \frac{8}{9} = -\frac{15}{18} = \boxed{-\frac{5}{6}}$$

b. (4 pts each) For which values of the number a do the following series converge (note the answers are different for each series).

i. $\sum_{k=1}^{\infty}\left(\frac{3}{a}\right)^k$

THIS IS A GEOMETRIC SERIES WITH RATIO $\frac{3}{a}$. IT CONVERGES FOR $\left|\frac{3}{a}\right| < 1$, that is FOR $\boxed{|a|>3}$

ii. $\sum_{k=1}^{\infty}\frac{a^k}{5^{2k}} = \sum_{K=1}^{\infty}\left(\frac{a}{25}\right)^K$

THIS IS A GEOMETRIC SERIES, SO IT CONVERGES FOR $\left|\frac{a}{25}\right| < 1$, THAT IS $\boxed{|a|<25}$

3. (15 pts) Let $f(x) = \sum_{k=1}^{\infty} \frac{(x-5)^k}{k \cdot 3^k}$.

a. (5 pts) Find the interval of convergence and radius of convergence for the above Taylor series (be sure to check convergence at endpoints if there are any).

USE THE RATIO TEST: $\lim_{k\to\infty} \dfrac{\frac{|x-5|^{k+1}}{(k+1)3^{k+1}}}{\frac{|x-5|^k}{k\cdot 3^k}} = \lim_{k\to\infty} \frac{k}{3(k+1)} \cdot |x-5| = \frac{|x-5|}{3}$

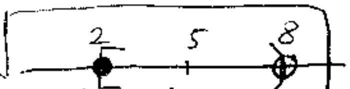
SO THE SERIES CONVERGES SURELY FOR $\frac{|x-5|}{3} < 1$, OR $|x-5| < 3$, THAT IS

$2 = 5-3 < x < 5+3 = 8$. $\rightsquigarrow$ $\boxed{\text{RADIUS OF CONVERGENCE } R = 3}$

CONVERGENCE AT ENDPOINTS:

$x = 2$: $\sum_{k=1}^{\infty} \frac{(-3)^k}{k\cdot 3^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$ CONVERGES (ALTERNATING SERIES)

$x = 8$: $\sum_{k=1}^{\infty} \frac{3^k}{k\cdot 3^k} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges (HARMONIC).



INTERVAL OF CONV: $[2, 8)$

b. (7 pts) Compute the Taylor series expansion about $x = 5$ for $g(x) = \int f(x)dx$, assuming that $g(5) = 0$. Write your answer using $\sum$ notation. In addition, explicitly write out the terms of degree ≤ 4 in this Taylor series (you don't need to simplify the coefficients when you write them down).

BY INTEGRATING TERMWISE:

$g(x) = \sum_{k=1}^{\infty} \frac{1}{k\cdot 3^k} \cdot \frac{(x-5)^{k+1}}{k+1} + C$. SET $x = 5$: $C = g(5) = 0$.

SO $g(x) = \frac{1}{3\cdot 2}(x-5)^2 + \frac{1}{2\cdot 3^2\cdot 3}(x-5)^3 + \frac{1}{3\cdot 3^3\cdot 4}(x-5)^4 + \ldots$

c. (3 pts) Find the interval of convergence for the series you computed in part **b**.

WE KNOW FROM THE GENERAL THEOREM THAT THE RADIUS OF CONVERGENCE FOR $g(x)$ is THE SAME AS THE RADIUS OF CONVERGENCE FOR $f(x)$, THAT IS $R = 3$. WE ONLY HAVE TO CHECK ENDPOINTS:

$x = 2$: $\sum_{k=1}^{\infty} \frac{(-3)^{k+1}}{k(k+1)\cdot 3^k} = \sum_{k=1}^{\infty} \frac{3(-1)^{k+1}}{k(k+1)}$ CONVERGES (alternating).

$x = 8$: $\sum_{k=1}^{\infty} \frac{3^{k+1}}{k(k+1)3^k} = \sum_{k=1}^{\infty} \frac{3}{k(k+1)}$ CONVERGES (BY APPLYING THE LIMIT COMPARISON TEST WITH THE p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$).

$\boxed{\text{SO THE INTERVAL OF CONVERGENCE IS } [2, 8]}$

4. (15 pts) Let $g(x) = \dfrac{e^x - 1}{x}$ for $x \neq 0$, and suppose that $g(0) = 1$.

a. (4 pts) Using the Taylor expansion for e^x about $x = 0$, show that

$$g(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}.$$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ SUBTRACT 1 AND DIVIDE BY X:

$$g(x) = \frac{e^x - 1}{x} = \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}$$

b. (4 pts) Use the series from part **a** to show that

$$g'(x) = \sum_{k=1}^{\infty} \frac{k}{(k+1)!} x^{k-1},$$

and check that this expansion for g' has infinite radius of convergence.

DIFFERENTIATE TERM BY TERM:

$$g(x) = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$$

$$g'(x) = \frac{1}{2!} + \frac{2x}{3!} + \frac{3x^2}{4!} + \frac{4x^3}{5!} + \cdots = \sum_{k=1}^{\infty} \frac{k\, x^{k-1}}{(k+1)!}$$

TO CHECK $R = \infty$, APPLY RATIO TEST:

$$\lim_{k\to\infty} \frac{\frac{(k+1)\,|x|^k}{(k+2)!}}{\frac{k\,|x|^{k-1}}{(k+1)!}} = \lim_{k\to\infty} \frac{k+1}{k(k+2)} \cdot |x| = 0 < 1, \text{ FOR ALL } X.$$

SO THE SERIES CONVERGES FOR ALL X, THAT IS $R = \infty$.

c. (5 pts) Explicitly compute $g'(x)$ using the definition of $g(x)$, and use this result to show that $g'(1) = 1$.

$$g(x) = \frac{e^x - 1}{x}$$

USE THE FORMULA $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$$g'(x) = \frac{e^x \cdot x - (e^x - 1)}{x^2} = \frac{e^x(x-1)+1}{x^2}$$

So $g'(1) = \frac{e^1(1-1)+1}{1} = 1.$

d. (2 pts) Explain how the results in parts **a**, **b** and **c** imply that

$$\sum_{k=1}^{\infty} \frac{k}{(k+1)!} = 1.$$

PLUG $x=1$ in THE TAYLOR SERIES FOR $g'(x)$:

$$g'(1) = \sum_{k=1}^{\infty} \frac{k}{(k+1)!}$$

SINCE $g'(1) = 1$, WE OBTAIN: $1 = \sum_{k=1}^{\infty} \frac{k}{(k+1)!}$

5. (14 pts) A power series $\sum_{k=0}^{\infty} a_k x^k$ centered at 0 converges for $x = -5$ and diverges for $x = 7$.

a. (2 pts each) Based on this information, determine what happens when x takes on the following values (you should answer either "converges," "diverges," or "cannot be determined from the information given" in each case). You don't need to provide any justification for your answers.

WE KNOW THAT THE RADIUS OF CONVERGENCE IS AT LEAST 5, AND AT MOST 7
SO WE KNOW FOR SURE ONLY THAT THE SERIES CONVERGES FOR $-5 \leq x < 5$, AND DIVERGES for $|x| > 7$

i. $x = 3$

CONVERGES

ii. $x = 6$

CANNOT BE DETERMINED

iii. $x = -10$

DIVERGES

iv. $x = 0$

CONVERGES

v. $x = -7$

CANNOT BE DETERMINED

b. (4 pts) Based on the information given, what is the smallest possible value for the radius of convergence for this power series? Be sure to justify your answer.

AS MENTIONED ABOVE, $5 \leq R \leq 7$. THIS IS BECAUSE THE SERIES DIVERGES FOR $|x| > R$ and CONERGES FOR $|x| < R$, BUT IT CAN BE EITHER CONV. OR DIV. FOR $x = \pm R$.

So the smallest possible value for the radius of convergence is 5

6. (14 pts) Antoinette wants to compute $\sqrt[3]{9}$, but she doesn't have a calculator with her. She decides to use a degree 2 Taylor polynomial for $f(x) = \sqrt[3]{x}$ centered at $x = 8$.

a. (6 pts) Compute the Taylor polynomial of degree 2 for $\sqrt[3]{x}$ centered at $x = 8$. Write your answer in the form $a_0 + a_1(x-8) + a_2(x-8)^2$, where a_0, a_1, a_2 are numbers which you should write as reduced form fractions (e.g. write $\frac{5}{2\cdot 3^2}$ or $\frac{5}{18}$ rather than $\frac{200}{6!}$).

2^{nd} Taylor POLYNOMIAL: $p_2(x) = f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2$

$f(8) = \sqrt[3]{8} = 2$, $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$, $f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$

$f'(8) = \frac{1}{3}\cdot 8^{-\frac{2}{3}} = \frac{1}{3}\cdot\frac{1}{8^{2/3}} = \frac{1}{3\cdot 4} = \frac{1}{12}$

$f''(8) = -\frac{2}{9}\cdot 8^{-\frac{5}{3}} = -\frac{2}{9}\cdot\frac{1}{8^{5/3}} = -\frac{2}{9}\cdot\frac{1}{32} = -\frac{1}{9\cdot 16}$

So $p_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{2\cdot 9\cdot 16}(x-8)^2$

b. (3 pts) Use your answer in part **a** to give an approximation to $\sqrt[3]{9}$. You do not need to simplify your answer in this case.

$\sqrt[3]{9} \simeq p_2(9) = 2 + \frac{1}{12} - \frac{1}{9\cdot 32}$

c. (5 pts) Show that the absolute value of the difference between the true value of $\sqrt[3]{9}$ and your answer in part **b** is $\leq .005 = \frac{1}{2}\cdot 10^{-2}$.

THIS WAS NOT DONE IN OUR CLASS,

but if you note the series continues as an alternating series with next term equal to

$+\frac{f'''(8)}{3!}(x-8)^3 = \frac{\frac{10}{27}(8)^{-8/3}}{3!}(x-8)^3$ which equals $\frac{\frac{10}{27\cdot 2^8}}{6}$

when $x=9$, so the difference is less than this,

or less than $\frac{10}{81\cdot 2^9} = \frac{5}{81\cdot 2^8} < \frac{5}{1000} = .005$

7. (15 pts) Determine the radius and interval of convergence for each of the following power series. Be sure to check convergence at the endpoints of the interval (if there are any endpoints).

a. (8 pts) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k^3} x^k$

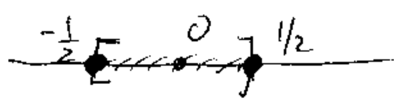
RATIO TEST: $\displaystyle\lim_{k\to\infty} \frac{\frac{2^{k+1}}{(k+1)^3}|x|^{k+1}}{\frac{2^k}{k^3}|x|^k} = \lim_{k\to\infty} \frac{2k^3}{(k+1)^3}|x| = 2|x|$ CONVERGES FOR $2|x|<1$, DIVERGES FOR $2|x|>1$.

SO THE RADIUS OF CONVERGENGE IS $R=\frac{1}{2}$

CHECK ENPOINTS: $x=\frac{1}{2}$: $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k^3}\cdot\frac{1}{2^k} = \sum_{k=1}^{\infty}\frac{1}{k^3}$ converges (p-series $p=3>1$)

$x=-\frac{1}{2}$: $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k^3}\left(-\frac{1}{2}\right)^k = \sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}$ converges (alternating).

SO THE INTERVAL OF CONVERGENCE IS $[-\frac{1}{2}, \frac{1}{2}]$



b. (7 pts) $\displaystyle\sum_{k=1}^{\infty} \frac{x^{3k}}{(k+3)!}$

RATIO TEST. $\displaystyle\lim_{k\to\infty} \frac{\frac{|x|^{3(k+1)}}{(k+1+3)!}}{\frac{|x|^{3k}}{(k+3)!}} = \lim_{k\to\infty} \frac{|x|^3}{k+4} = 0 < 1$

THE SERIES CONVERGES FOR ALL X, $R=\infty$.