Math 1B: Final Exam

Harvard University
Friday, 21 January 2000, 2:15 – 5:15 pm

Name ____

Circle the name of your section head:

Peter Clark MWF 10	Andy Engelward TTh 11:30	Tammy Lefcourt MWF 11
Yu-ru Liu TTh 10	Curt McMullen MWF 10	Alexandru Popa TTh 11:30
Kiril Selverov MWF 12	Greg Warrington MWF 11	Yuhan Zha TTh 10
	Nina Zipser TTh 10	

- Justify your work. The clarity and completeness of your presentation will also count towards your grade.
- This is a *closed book* exam. Do not refer to the text or any other written materials. Calculators are not permitted.

	Points	Score
1	9	
2	7	
3	10	
4	6	
5	8	
6	8	
7	8	
8	8	
9	10	
10	10	
11	9	
12	7	
Total	100	

1. Determine if the following series converge or diverge. **Justify your answers.**

   (a) (3 points) $\sum_{n=2}^{\infty} \frac{n^5}{5^n \ln n}$

   (b) (3 points) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

   (c) (3 points) $\sum_{n=1}^{\infty} (-1)^n n^{-1/2}$

2. (7 points) What is the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n+n^2}}$? (Be sure to check convergence at the endpoints.)

3. Evaluate the following integrals.

(a) (5 points) $\int x^{1/2} \ln x \, dx$

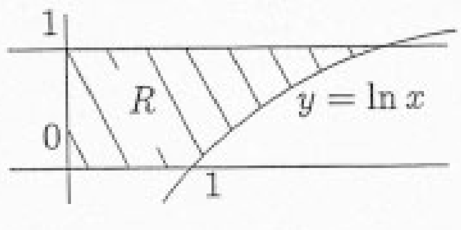
(b) (5 points) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}(e^{\sqrt{x}}+1)} \, dx$

4. (6 points) Is the improper integral

$$\int_{-7}^{8} \frac{dx}{(x-2)(x+1)}$$

convergent or divergent? Justify your answer.

5. (8 points) Let R be the region bounded by the graph of $y = \ln x$ and the lines $y = 0$, $y = 1$ and $x = 0$.



Find the volume of the solid obtained by revolving R about the y-axis.

6. (8 points) The Maclaurin series for $\tan x$ is given by:

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

Using the series, find the first 3 nonzero terms in the Maclaurin series for $f(x) = \ln(\cos x)$. (Hint: what is $f'(x)$?)

7. (8 points) Find the length of the arc of the curve $x^{2/3} + y^{2/3} = 1$ between $x = 0$ and $x = 1$ and above the x-axis.

8. (8 points) Solve the differential equation $\dfrac{dy}{dx} = x + xy$, subject to the initial condition $y(0) = 3$.

9. Consider a 10-ounce cup full of black coffee. At time $t = 0$ we begin adding cream at the rate of 5 ounces per minute, while stirring. As cream is added, the cup overflows, so its total contents remains 10 ounces.

   (a) (3 points) Write down a differential equation for the amount of cream $y(t)$ in the cup after t minutes of pouring.

   (b) (1 point) What is $y(0)$?

   (c) (5 points) Solve for $y(t)$ with the given initial condition.

   (d) (1 point) How much cream is in the cup after $t = 2\ln 2$ minutes?

10. (a) (5 points) Consider a weighted spring whose motion is governed by the differential equation

$$\frac{d^2x}{dt^2} + 9x = 0.$$

Solve for $x(t)$, given that $x(0) = 2$ and $x'(0) = 0$.

(b) (2 points) Graph $x(t)$, and describe in words the motion of the spring over time.

(c) (3 points) Now suppose the spring is immersed in molasses so it becomes critically damped. Write down (but do not solve) the differential equation governing its motion now.

11. Consider the differential equation $\frac{dy}{dt} = \sin(y)$.

(a) (5 points) Sketch the solution curves for y values between 0 and 3π. Show all equilibrium (constant) solutions, as well as the solution curves for $y(0) = 0.1$, 6 and 7.

(b) (2 points) Which equilibrium solutions between 0 and 3π are stable? Which are unstable?

(c) (2 points) Suppose $y(0) = 0.1$ What is the limit of $y(t)$ as $t \to +\infty$?

12. (7 points) Consider the differential equation $y'' = xy$. Find the first three nonzero terms of the power series solution with $y(0) = 1$, $y'(0) = 0$.