**solutions**

1. The integral converges precisely when $p > 1$: indeed, if $p = 1$, $\int_1^\infty \frac{dx}{x} = \ln x|_1^\infty = \ln\infty - \ln 1 = \infty$, divergent. If $p \neq 1$, we have $\int_0^\infty \frac{dx}{x} = \frac{x^{1-p}}{1-p}|_1^\infty = \frac{\infty^{1-p}}{1-p} - \frac{1}{1-p}$. For this to converge we need $1 - p < 0$, i.e. $p > 1$.

2. It converges precisely when $p < 1$: indeed, if $p = 1$, $\int_0^1 \frac{dx}{x} = \ln x|_0^1 = \ln 1 - \ln 0 = 0 - (-\infty) = \infty$, divergent. If $p \neq 1$, $\int_0^1 \frac{dx}{x^p} = \frac{x^{1-p}}{1-p}|_0^1 = \frac{1}{1-p} - \frac{0^{1-p}}{1-p}$; for $0^{1-p}$ to be finite, we need $1 - p > 0$, i.e. $p < 1$.

3. In view of problems 1 and 2, never.

4. First we must recognize that it is improper. Indeed, the way to try to do the integral anyway is by partial fractions, so $\frac{1}{x^2-3} = \frac{1}{(x-\sqrt{3})(x+\sqrt{3})} = \frac{A}{x-\sqrt{3}} + \frac{B}{x+\sqrt{3}}$, and solving for $A, B$ we get $\frac{1}{x^2-3} = \frac{\frac{1}{2\sqrt{3}}}{x-\sqrt{3}} - \frac{\frac{1}{2\sqrt{3}}}{x+\sqrt{3}}$. So the function becomes infinite at $\sqrt{3}$, which is in the interval of integration. Proceeding as in the second problem, we get that the interval is divergent – indeed, the integral of any rational function $\int_a^b \frac{P(x)}{(x-a)^n Q(x)}$ (with $P(a) \neq 0$) will be divergent.

5. Notice that it looks hopeless to find an antiderivative for $e^{-x^3}$, so we cannot evaluate the integral exactly. Nevertheless, the integrand goes to zero very rapidly, so we intuitively expect the integral to converge. To make this idea work, we make a comparison: when $x \geq 1$, then $x^3 > x$, so $e^{x^3} > e^x$ and $e^{-x^3} < e^{-x}$. Thus $\int_1^\infty e^{-x^3} dx \leq \int_1^\infty e^{-x} dx = -e^{-x}|_1^\infty = -(e^{-\infty} - 1) = 1$. So the integral converges (and is less than 1).

6. If we split the integral into two pieces, a finite piece and an infinite piece we can handle each piece separately. Say $\int_1^\infty e^{-x^3} dx = \int_1^{10} \infty e^{-x^3} dx + \int_{10}^\infty e^{-x^3} dx$. The latter term is very small: $\int_{10}^\infty e^{-x^3} dx \leq \int_{10}^\infty e^{-x} dx = e^{-10} < .00005$ (using a calculator). Thus we are left to approximate $\int_1^{10} e^{-x^3} dx$, and we could use any of our numerical techniques, e.g. the midpoint approximation. We leave the calculation to the reader.

7. Since $|\frac{\sin x}{x^2}| \leq \frac{1}{x^2}$, and from the first problem $\int_1^\infty \frac{dx}{x^2} < \infty$, the given improper integral is *absolutely convergent.* (The textbook does not discuss absolute convergence of improper integrals, but the results are the same as for infinite series.)

8. This time the improper integral is *not* absolutely convergent. Indeed, the function is analogous to an alternating series – the function $\frac{\sin x}{x}$ is bounded by $+/-\frac{1}{x}$ and changes sign at integer multiples of $\pi$. Looking at its graph, we see that the signed areas alternate between being positive and negative, and the unsigned areas decrease to zero in size. Therefore

the alternating series test applies to the alternating sum of the areas, which is what the improper integral represents. From this we see that the integral converges. (This was a problem on the Fall 1999 qualifying exam for Harvard math grad students!)

9. Let $A(x)$ denote the area of the semicircle at $x$. Then the volume is $V = \int_0^9 A(x)dx$. To find the area, note that the radius of the semicircle is $\sqrt{x}$, so that the area is $\frac{1}{2}\pi R^2 = \frac{1}{2}\pi x$. Thus the volume is $\frac{\pi}{2}\int_0^9 x dx = \frac{81\pi}{4}$.

10. If we think of the sphere as the solid region whose cross-sections are circles on the bae $x^2 + y^2 = R^2$, then the volume is $\int_{-R}^{R} A(x)dx = \int_{-R}^{R} \pi(\sqrt{R^2 - x^2})^2 dx = \pi \int_{-R}^{R} (R^2 - x^2)dx$, which indeed evaluates to $\frac{4}{3}\pi R^3$.

11. The volume is $\frac{\pi}{3}(r^2 h + R^2 h + Rrh)$.

12. When we use the method of washers, the variable of integration is the variable which is *perpendicular* to the axis of rotation: if we revolve around something parallel to the $y$-axis we integrate with respect to $x$, while if we revolve around something parallel to the $x$-axis we integrate with respect to $y$. When we use the method of shells, the variable of integration is the variable which is *parallel* to the axis of integration: if we revolve around a line parallel to the $x$-axis we integrate with respect to $x$, while if we revolve around a line parallel to the $y$-axis we integrate with respect to $y$. Thus we are always able to choose the variable of integration *or* the method of integration. You might think that we would prefer to always integrate with respect to $x$, but this need not be true: sometimes the problem is easier if we integrate with respect to $y$. E.g., see the next problem.

13. The first thing to do is to draw a picture showing the two points of intersection of the parabola and the line. To find those points analytically, we set $y^2 = x = 2y + 3$ and solve to get $y = 3$ or $y = -1$. Thus the two intersection points are $(1, -1)$ and $(9, 3)$. Notice that if we integrate with respect to $x$ (and use washers) we would have to break up the integral into two pieces: from $x = 0$ to 1 the parabola is the bottom function, while from $x = 1$ to 9 the line is the bottom function. But notice that it is always the case that the line is farther to the right of the parabola, meaning that if we integrate with respect to $y$ (and use shells) we can do the integral all at once. The integral we get is $2\pi \int_{-1}^{3} y[(2y+3) - y^2]dy = \frac{32\pi}{3}$.

14. If we do the problem using washers, the integral we get is $\pi \int_0^1 (1-x)^2 - (1-\sqrt{x})^2 dx$, whereas if we do the problem using shells, the integral we get is $2\pi \int_0^1 y(y - y^2)dy$. Either way the final answer is $\frac{\pi}{6}$.

15. We compute that $f'(x) = \frac{\sec x \tan x}{\sec x} = \tan x$, so the arc length is $\int_0^{\pi/4} \sqrt{1 + \tan^2 x}dx = \int_0^{\pi/4} \sec x dx = \ln|\sec x + \tan x|_0^{\pi/4} = \ln|\frac{2}{\sqrt{2}} + 1| - \ln|1 + 0| = \ln(\sqrt{2} + 1)$.

16. Let's try to show that $\int_c^d (mx+b)dx$ is equal to $f(\frac{c+d}{2})(d-c)$. We have $\int_c^d (mx+b)dx = mx^2/2 + bx|_c^d = m(d^2/2 - c^2/2) + b(d-c) =$ $(d-c)(\frac{m}{2}(c+d)+b) = f(\frac{c+d}{2})(d-c)$.

17. The point is that you have to pull the chain up as you go, so that at the bottom the force includes the total weight of the chain, 10 kg, whereas at the top the chain does not contribute at all. Writing the force as a function of a coordinate $y$ ranging from 0 to 10, we get $F(y) = (15 + (10-y))g = (25-y)g$ (here $g$ is the acceleration due to gravity, which is approximately 9.8, but we'll just leave it as $g$). So the work is $W = \int_0^1 0(25-y)gdy = 250 - \frac{y^2}{2}g|_0^{10} = 200g$ newtons. For a slicker solution, notice that since the force contributed by the chain depends linearly on the height, using the previous problem we know that the average force occurs in the middle – it is $5g$. The work is the distance times the average force, so is $10(15g + 5g = 200g$, the correct answer.

18. If we coordinatize the depth of the water using a coordinate $y$ that runs from $-10$ to 10, then the amount the water must be pumped as a function of $y$ is $d(y) = 10 - y$, whereas the cross-sectional area at $y$ is $A(y) = \pi(100 - y^2)$. Since the general formula for the work done in this manner is $W = \int_a^b \rho g A(y) d(y) dy$, our setup is $\int_{-10}^{10} \rho g \pi (100 - y^2)(10-y)dy = \frac{\rho g \pi 40{,}000}{3} = \frac{9{,}810\pi(40{,}000)}{3}$, a heck of a lot of work.

19. The spherical tank is completely symmetrical about the plane $y = 0$: for every point inside the sphere with a positive $y$-coordinate that must be pumped less than 10 meters, there is a point with the same negative $y$-coordinate that must be pumped the corresponding amount more than 10 meters. Therefore, we reason that the final answer must be the same as if all the water were being pumped 10 meters, i.e. the work done must be 10 meters times the total weight of the water, so $W = 10(\frac{4}{3}\pi R^3 1000g) = \frac{4}{3}\pi(10)(1000)(9810g) = \frac{9{,}810\pi(40{,}000)}{3}$, the exact same answer. (This technique of using symmetry to circumvent the calculus is one familiar to anyone who has taken a "physics without calculus" course. Just because you know how to integrate doesn't mean that you should forget to think about the situation geometrically – sometimes "elementary" geometric reasoning leasds to a complete solution.

20. Let's check that the second derivative is positive. Indeed, $f'(x) = 2xe^{x^2}$, so $f''(x) = 2e^{x^2} + 4x^2e^{x^2} = (2+4x^2)e^{x^2}$, and indeed both terms in the product are positive for all real $x$. Now the point of the problem is to remind you that when a function has constant concavity, the midpoint tangent method and the trapezoidal method give actual bounds for the integral: in our concave up case, the tangent line always lies below the function and the secant line always lies above the function, so any midpoint approximation will be a lower bound and any trapezoidal approximation will be an upper bound. Since no degree of accuracy was specified, let's only divide into

one subinterval: we get that the midpoint approximation is $\frac{e^{(\frac{1}{2})^2}}{1-0} = e^{\frac{1}{4}}$, and the trapezoidal approximation is $\frac{e^0+e^1}{2(1-0)} = \frac{1+e}{2}$. Thus we can say that $e^{\frac{1}{4}} < \int_0^1 e^{x^2}\,dx < \frac{1+e}{2}$. (This is not a very sharp approximation, but by increasing $n$ we could bound the integral to any desired degree of accuracy.)