Harvard University
Math 1b. Second Semester Calculus
Final Exam

Name ______________________

January 25, 1999

Final Exam

- Do not open this exam booklet until you are directed to do so.
- You have 3 hours to earn 100 points.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Do not put part of the answer to one problem on the back of the sheet for another problem.
- Do not spend too much time on any one problem. Read them all through first and work on them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat. By the same token, be sure to justify your solutions (unless you are explicitly told otherwise), so we can follow your reasoning.
- Good luck!

Problem	Points	Grade
1	10	
2	10	
3	10	
4	12	
5	10	
6	7	
7	10	
8	13	
9	10	
10	8	
Total	100	

Please circle your section:

MWF 10:00	MWF 10:00	MWF 10:00	MWF 11:00	MWF 11:00	MWF 12:00
Brian Conrad	Andy Engelward	Eric Towne	Noam Elkies	Robert Pollack	Andy Engelward

TTh 10:00	TTh 10:00	TTh 10:00	TTh 11:30	TTh 11:30
Tomas Klenke	Joel Rosenberg	Eric Towne	Heather Russell	Nadia Lapusta

1. (10 pts)

i. (2 pts each) For each of the following, circle the correct answer. No justification is required.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{\left(10^{(10^{10})}\right)^n}{(2n+1)!}$ converges diverges

b. $\displaystyle\sum_{n=1}^{\infty} \left(30 - \frac{1}{n^2}\right)$ converges diverges

c. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+10}}$ converges diverges

ii. (4 pts) Does $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^3+1} = \frac{1}{1^3+1} + \frac{2}{2^3+1} + \frac{3}{3^3+1} + \frac{4}{4^3+1} + \dots$ converge? If so, why? If not, why not?

2. (10 pts) Due to the bizarre mating rituals of the three-legged, two-tongued, yellow-eyed toads of the South Bernoulli Islands, their population at time t, $P(t)$, satisfies the following differential equation:

$$\frac{dP}{dt} = -7P(P-1000)(P-100000).$$

a. (4 pts) Find the equilibrium solutions and determine for each whether it is stable or unstable.

b. (3 pts) Is there some positive number so that if the population is ever below this number, then the species is doomed to extinction? Explain why or why not.

c. (3 pts) Is there some number so that if the population is higher than this number, then the population will grow without bound? Explain why or why not.

3. (10 pts) Suppose r is a positive constant. Consider the semicircle given by $x^2 + y^2 = r^2$ with $y \geq 0$ and $-r \leq x \leq r$. If we rotate this semicircle about the x-axis, we obtain a sphere with radius r.

a. (6 pts) Using integration, compute the surface area of this sphere. Your answer should depend on r.

b. (4 pts) Using integration. compute the volume of this sphere. Your answer should depend on r.

4. (12 pts) Evaluate the following integrals.

a. (2 pts) $\int \ln x \, dx$

b. (4 pts) $\int \frac{dx}{x^3 - x}$

c. (2 pts) $\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx$

d. (4 pts) $\int 3x^5 \sqrt{1+x^3} \, dx$

5. (10 pts)

a. (3 pts) Finds the terms of degree ≤ 4 in the Taylor series about $x = 0$ (i.e., in the Maclaurin series) of the function

$$f(x) = \sin x + \cos x.$$

b. (1 pt) Use your answer from part **a** to obtain an estimate for

$$\sin(\tfrac{1}{10}) + \cos(\tfrac{1}{10}).$$

(You need not simplify your answer.)

c. (2 pts) Compute the 5th derivative $f^{(5)}$ of f and explain why

$$|f^{(5)}(x)| \leq 2$$

for all x.

d. (4 pts) Using the fact that $|f^{(5)}(x)| \leq 2$ for all x (even if you were unable to verify this in **c**), show that your estimate in **b** differs from the true value by at most

$$2 \cdot 10^{-7}.$$

6. (7 pts)

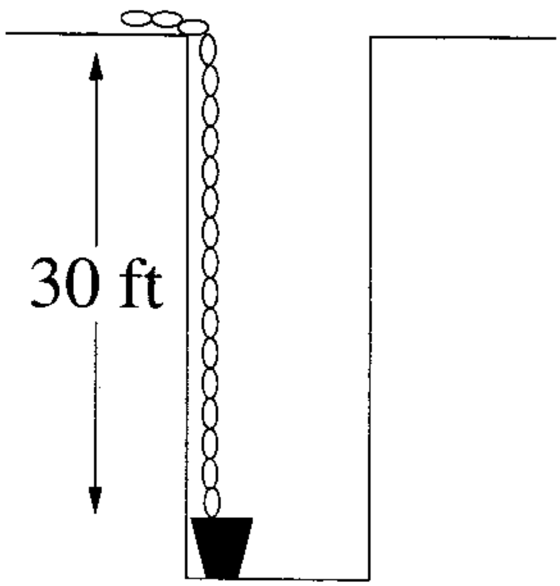
a. (5 pts) Find the general solution of the differential equation

$$5y + xy' = x^2$$

with $x > 0$.

b. (2 pts) Check that your answer is in fact a solution of this differential equation.

7. (10 pts) Brian raises a bucket of calculus books from the bottom of a well, by pulling up on a chain attached to the bucket (see picture). The chain measures 30 feet from the top of the well to the bucket. The chain weighs 2 pounds per foot. The bucket and the books together weigh 100 pounds. If Brian pulls both the bucket (with the books) and the whole length of chain all the way up to the top of the well, then calculate the amount of work that he has done. (You may ignore the size, but not the weight, of the bucket and the books.)



8. (13 pts) A 1000 gallon pool initially contains 500 gallons of water with 1500 milligrams of a chemical mixed in. A solution containing 4 milligrams of chemical per gallon of water enters at a rate of 2 gallons per minute, while the well-mixed liquid flows out of the pool at a rate of 2 gallons per minute. Let $C(t)$ denote the amount (in milligrams) of the chemical in the pool after t minutes of time have passed.

a. (4 pts) Write a differential equation whose solution is $C(t)$, and give the value of $C(0)$.

b. (4 pts) Solve for $C(t)$. Your solution should have no undetermined constants.

c. (2 pts) What happens to $C(t)$ as $t \to \infty$?

d. (3 pts) Suppose that the situation is exactly as described above, except that the inflow rate is 3 gallons per minute (and the outflow rate remains at 2 gallons per minute). Write a differential equation whose solution is $C(t)$ in this situation. You do NOT need to solve this differential equation.

9. (10 pts)

a. (3 pts) Find the terms of degree ≤ 6 in the Taylor series about $x = 0$ for the function $e^{x^2} - \sin(x^2) - 1$.

b. (7 pts) Let $f(x) = \frac{1}{1+4x}$.

i. (2 pts) Which of the following infinite series is equal to the Taylor expansion of f about $x = 0$? You do not need to justify your choice.

$$\text{(I)} \sum_{n=0}^{\infty}(4x)^n \quad \text{(II)}\ 4 \cdot \sum_{n=0}^{\infty} x^n \quad \text{(III)} \sum_{n=0}^{\infty}(-4)^n x^n \quad \text{(IV)}\ \frac{1}{4}\sum_{n=0}^{\infty}(-x)^n$$

ii. (5 pts) For your choice above in **i**, compute the radius and interval of convergence.

10. (8 pts) A power series

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

satisfies the differential equation $y'' + xy = x$, $y(0) = 6$, $y'(0) = 12$. Write out the terms of degree ≤ 6.

Your answer should be in the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6,$$

where the coefficients $a_0, \ldots, a_6$, are numbers which you must compute. (You do not need to write terms with a coefficient of 0.)