Harvard University
Math 1b. Second Semester Calculus
Final Exam

Name ______________________

May 21, 1999

Final Exam

- Do not open this exam booklet until you are directed to do so.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Do not put part of the answer to one problem on the back of the sheet for another problem.
- Do not spend too much time on any one problem. Read them all through first and work on them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat. By the same token, be sure to justify your solutions (unless you are explicitly told otherwise), so we can follow your reasoning.
- Good luck!

Problem	Points	Grade
1	10	
2	10	
3	10	
4	9	
5	10	
6	9	
7	10	
8	10	
9	10	
10	12	
Total	100	

Please circle your section:

MWF 10:00	MWF 11:00	MWF 12:00	TTh 10:00	TTh 11:30
Brian Conrad	Grisha Mikhalkin	Cathy O'Neil	Andy Engelward	Andy Engelward

1. (10 pts) For each of the three infinite series below, determine whether it converges or diverges and explain why.

a. (2 pts)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n}+10}$$

b. (3 pts)

$$\sum_{k=1}^{\infty} \frac{k+2}{k!}$$

c. (3 pts)

$$\sum_{n=1}^{\infty} e^{-n}$$

d. (2 pts) The great 18th century mathematician Euler discovered the remarkable formulas

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Using these facts, determine the value of

$$\sum_{n=1}^{\infty} \frac{\pi^2 n^3 - 15n}{n^5}.$$

2. (10 pts) James Bond's unsophisticated cousin Jack is drinking a shaken and stirred martini from a glass in the shape of the surface of revolution obtained from rotating the parabola $y = x^2$ around the y-axis. There is no cherry or ice or other non-fluid substance in the drink. The glass is filled with fluid up to a height of a inches, as shown.

a. (4 pts) Compute the volume $V(a)$ (in cubic inches) of the fluid in the glass.

b. (6 pts) Compute the surface area $S(a)$ (in square inches) of the part of the glass which touches the fluid.

3. (10 pts) Consider the second-order differential equation

$$y'' - 3y' + 2y = 0.$$

a. (10 pts) Find the terms of degree ≤ 4 (i.e. the first 5 coefficients) in the MacLaurin series of the solution to the above differential equation with initial conditions $y(0) = 1$, $y'(0) = 2$.

b. (1 pt) **Extra Credit**. Based upon your answer above, try to guess the exact solution.

4. (9 pts) **a.** (3 pts) Compute the antiderivative

$$\int \frac{dx}{x\ln(x)}$$

b. (1 pt) Determine if your answer in **a** is correct by computing its derivative.

c. (3 pts) Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)dx$$

d. (2 pts) Does the integral

$$\int_{-1}^{1} \frac{dx}{x^4}$$

converge? Justify your answer.

5. (10 pts) Consider the differential equation

$$\frac{dy}{dx} - y = e^x.$$

a. (7 pts) Find the general solution to the above differential equation. Your answer should involve an undetermined constant.

b. (3 pts) Check if your general solution satisfies the given differential equation (you must check your *general* solution, and not just the solution with one choice of a value for the undetermined constant).

6. (9 pts) Find the interval and radius of convergence for each of the following infinite series. Don't forget to check convergence at the endpoints of the interval (if the endpoints are not $\pm\infty$).

a. (3 pts)

$$\sum_{k=1}^{\infty} \frac{3^k x^k}{\sqrt{k}}$$

b. (3 pts)

$$\sum_{k=1}^{\infty} \frac{k^2 (x-1)^k}{2^k}$$

c. (3 pts)

$$\sum_{k=1}^{\infty} \frac{5^k (x - \frac{1}{2})^{2k}}{(k+2)!}$$

7. (10 pts) Sam has an 80 gallon fish tank. The water in the tank has 2 lbs of chlorine dissolved in it, which is too much to be safe for the fish. Beginning at noon, Sam runs water containing $\frac{1}{100}$lb of chlorine per gallon into the tank at a rate of 2 gallons per minute, while also draining off water from the tank at the same rate.

a. (3 pts) What differential equation governs the number $c(t)$ of pounds of chlorine after t minutes (so $t = 0$ represents noon)? Also write down the initial condition.

b. (6 pts) Solve the differential equation with initial condition which you determined in part **a**.

c. (1 pt.) What happens to your answer from part **b** as $t \to \infty$?

8. (10 pts)

a. (6 pts) Compute the terms of degree ≤ 4 in the MacLaurin series of

$$f(x) = (1+x)^{1/4}.$$

Write your answer in the form $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ where $a_0, \ldots, a_4$ are numbers which you must determine.

b. (4 pts) Using the identity

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

and the known MacLaurin series for the cosine function, compute the terms of degree ≤ 4 in the MacLaurin series for $\sin^2(x)$. Write your answer in the form $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ where $a_0, \ldots, a_4$ are numbers which you must determine.

9. (10 pts) According to Newton's Law of Gravity, a satellite orbiting the earth at a distance of r miles from the center of the earth feels a gravitational force of $F = k/r^2$, where k is a positive constant which depends upon the mass of the satellite.

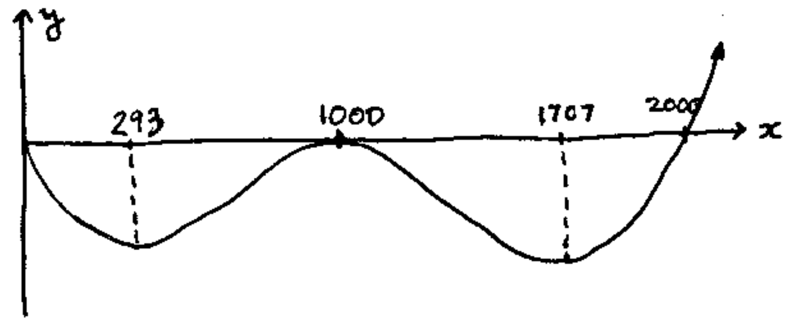
a. (6 pts) A satellite is presently at a distance of 5000 miles from the center of the earth, but the owner of the satellite wants to place it in an orbit 6000 miles from the center of the earth. In terms of the above constant k which is associated with the satellite, compute the work against gravity which is required to move the satellite from its present position to the desired higher orbit.

b. (4 pts) Suppose the same satellite is now at an orbit 6000 miles from the center of the earth and the owner wants to use it as space probe. By using a suitable improper integral, compute (again in terms of k) the amount of work against the earth's gravity which is required to send the satellite "off to infinity." It is a marvelous fact that the answer is finite!

10. (12 pts) The Millenium Falcon, widely believed to be the fastest spaceship in the galaxy ever since it outran several Imperial Star Destroyers, is trying to fly past the Death Star without getting sucked in by the Death Star's powerful tractor beam. The distance $x(t)$ between the spaceship and the Death Star at time t satisfies the differential equation

$$\frac{dx}{dt} = x(x-1000)^2(x-2000),$$

where the graph of $y = x(x-1000)^2(x-2000)$ is given approximately by the following:



a. (8 pts) Sketch the equilibrium lines, and also sketch the solution curves with initial values $x(0) = 125, 750, 1800, 2300$ (you do not need to label the equilibrium lines as stable or unstable).

b. (4 pts) For which initial values $x(0) \geq 0$ does the spaceship manage to escape from the region near the Death Star (do not just consider the four initial values in **a**)? Justify your answer.

c. (0 pts) Which of the "Star Wars" movies is the best? Which is the worst? Justify your answer.