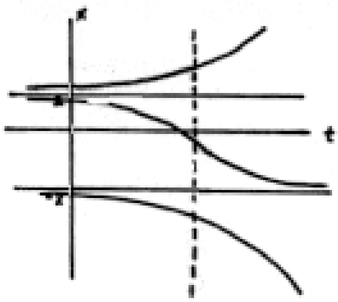
Solutions to Math 18 Exam III
Dec. 10, 1990

The problems on this part of the exam are multiple choice and short answer problems. We [u]rge you to use your __understanding__ of the material to answer these questions with a minimal [amount] of computation.

....

[1. (5] points) Below is the graph of several particular solutions to a differential equation:



By visual inspection, we see...

... that $\frac{dx}{dt} = 0$ when $x = 1$ or $x = -2$,

... that $\frac{dx}{dt} > 0$ when $x > 1$, and

... that $\frac{dx}{dt} \leqslant 0$ when $x < 1$.

[W]hich one of the following differential equations could have the solutions pictured above?

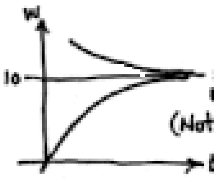
[(a)] $\frac{dx}{dt} = -(t-1)(t+2)$ (e) $\frac{dx}{dt} = x(x-1)(x+2) \gtrless 0$ when $x \lessgtr 0$.... No good.

[(b)] $\frac{dx}{dt} = (t-1)(t+2)$ (f) $\frac{dx}{dt} = (x-1)(x+2) > 0$ when $x < -2$ ...... No good.

[(c)] $\frac{dx}{dt} = (t-1)^2(t+2)$ (g) $\frac{dx}{dt} = (x-1)^2(x+2) > 0$ when $x = 0$...... No good.

[(d)] $\frac{dx}{dt} = (t-1)(t+2)^2$ (h) $\frac{dx}{dt} = (x-1)(x+2)^2$ ⟸ JUST RIGHT!!

None of these could be right because they are equations suggesting that the slope of a solution curve is dependent only on the horizontal coordinate, t. In the diagram given, there are several solutions drawn, each of which has a __different__ slope through the vertical dashed line (i.e., at the __same__ t value, these curves have __different__ slopes).



(Notice that $\frac{dw}{dt} < 0$ for $w > 10$ and $\frac{dw}{dt}$ ...

2. (5 points) Which __one__ of the following differential equations has a __stable__ equi[librium] ...

(a) $\frac{dw}{dt} = w - 10$

(b) $\frac{dw}{dt} = 10 - w$

(c) $\frac{dw}{dt} = (w-10)^2$

(d) $\frac{dw}{dt} = -(w-10)^2$

(e) $\frac{dw}{dt} = t - 10$

(f) $\frac{dw}{dt} = 10 - t$

(g) $\frac{dw}{dt} = (t-10)^2$

(h) $\frac{dw}{dt} = -(t-10)^2$

w=10 is not o[...] to __any__ of the[...] Indeed, none [...] equations ha[...] solutions. (T[...] different for [...] so the solutio[...] constant.)

3. (5 points) The trajectories drawn below are solutions to which __one__ of the fo[llowing] equations?

(a) $\frac{ds}{dt} = -s^2$ (dependent on s only)

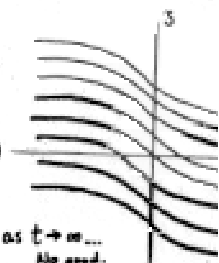
(b) $\frac{ds}{dt} = -s$ (dependent on s only)

(c) $\frac{ds}{dt} = -e^{-t^2}$ $\begin{cases}\text{always negative,} \\ \to 0 \text{ as } |t| \to \infty\end{cases}$

(d) $\frac{ds}{dt} = (1-s)s$ (dependent on s only)

(e) $\frac{ds}{dt} = -t > 0$ when $t < 0$... No good.

(f) $\frac{ds}{dt} = -3te^{t^2} \to \infty$ in absolute value as $t \to \infty$... No good.



Notice, if the right-hand side were dependent only on s, then there would be constant solutions should the right-hand side ever be zero. Thus, (a), (b), and (d) are quickly disposed of

Notice that the slope of the solu[tion] depends only on the t-coordinat[e ...] any __fixed__ value of t, all the s[...] curves have the same slope. For [...] when t=0, it looks like $\frac{ds}{dt}$ [...]

cafeteria is preparing a large batch of hot chocolate. The chef is stirring a mixture t chocolate and 1 cup warm milk in a large pot when her assistant begins to add a ch is 90% hot chocolate and 10% warm milk. The addition is made continuously at ups per minute. As the assistant begins this addition, the mixture begins to be used 2 cups per minute.

e a differential equation whose solution is $M(t)$ = the number of cups of milk in the t. Set up the differential equation but <u>don't solve it</u>.

rate in rate out

Equation: $\frac{dM}{dt} = .2 - \frac{2M}{5}$ (See below)

pose the scenario in part (a) is changed to the following: When the assistant begins mixture of 90% hot chocolate, 10% milk at the 2 cups/min. rate (and when the draining at the 2 cup/min. rate), a second assistant begins to pour in coffee at a $\frac{1}{2}$ te. Please write a differential equation whose solution is $M(t)$ the number of cups of pot at time t. Do not solve your equation.

rate in rate out

Equation: $\frac{dM}{dt} = .2 - \frac{2M}{5 + t/2}$ (See below)

0% choc.}
0% milk}

2 cups/min.

ial
lume is
tant : 5 cups

Rate of Milk "<u>in</u>" : $(.10)(2 \text{ cups/min}) = .2$ cups/min

Rate of Milk "<u>out</u>" : $\underbrace{\left(\frac{M(t)}{5 \text{ cups}}\right)}_{\boxed{\text{Milk concentration}}}(2 \text{ cups/min}) = \frac{2M}{5}$ /min.

(units of M are "cups")

(B) Adding $\frac{1}{2}$ cup of coffee/min. doesn't change the rate at which milk enters vat.

But the volume is now changing: $V(t) = (5 + t/2)$ gals

The concentration of milk is : $\frac{M(t)}{V(t)} = \frac{M(t)}{5 + t/2}$

Thus, rate out is : $\left(\frac{M}{5 + t/2}\right)(2 \text{ cups/min.})$

$\frac{dy}{dx} = x^2$

$y = \frac{x^3}{3} + C$

(And $dy/dx > 0$)

(And notice th[…]
when $y > 0$

ELL[…]

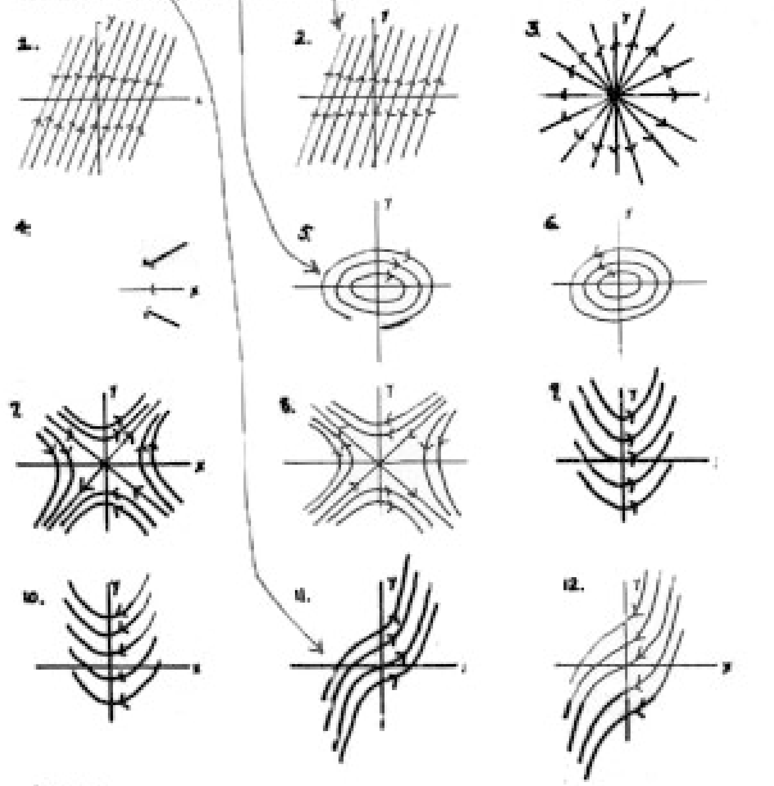
$\frac{dy}{dx} = \frac{-2x}{y} \Rightarrow y\,dy = -2x\,dx \Rightarrow \frac{y^2}{2} = -x^2 +$

5. **(9 points)** Consider the following systems of differential equations:

(a) $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = x^2$ (b) $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -2x$ (c) $\frac{dx}{dt} = y$, $\frac{dy}{dt} = 3y$ $\Big\} \to \frac{dy}{dx} = \frac{3y}{y} = 3$ (Slope is constant :

(Notice $\frac{dx}{dt} > 0$ when

Match each of the pairs of differential equations with the correct figure below.



Answer:

Graph # 11 corresponds to system (a).

Graph # 5 corresponds to system (b).

Graph # 2 corresponds to system (c).

u must show all your work.

....

nts) Since the early 1960's the population of the crown-of-thorns starfish on Australia's Barrier Reef has been increasing dramatically. This has been causing alarm, since it feeds ng coral. A hungry crown-of-thorns can eat its way through 5 square meters of coral in a

) If the number of crown-of-thorns in the Barrier Reef area was 40 in 1960 and 160 in find $s(t)$, the number of crown-of-thorns starfish in the Barrier Reef area at time t, where corresponds to the year 1960. Note: We can assume that the crown-of-thorns population ses at a rate proportional to itself. (The crown-of-thorns' only natural enemy is the triton, e triton doesn't depend on the crown-of- thorns for food and there are very few tritons in ea.) so we ignore their impact.

= the number of starfish at time t.

kS

$k\,dt$

$kt + C_1$

e

e^{kt}

$=0,\ S=40$

$Ce^0 = C = 40$

$40e^{kt}$

At $t=9$, $S=160$

$160 = 40e^{9k}$

$4 = e^{9k}$

$\ln 4 = 9k$

$k = \frac{\ln 4}{9}$

$$\boxed{S = 40e^{\frac{\ln 4}{9}t}} = 40\left(e^{\frac{1}{9}\ln 4}\right)^t = 40\left(e^{\ln 4^{1/9}}\right)^t = 40\left(4^{t/9}\right)$$

b) Assuming all crown-of-thorns starfish are hungry, how many square meters of cdal have destroyed by these starfish between 1960 and 1990?

Partition [0,30] into n equal pieces

30

# of starfish $= 40e^{\frac{\ln 4}{9}t_i} = 40\cdot 4^{t_i/9}$

in the i^{th} interval $(5\Delta t\ \underbrace{40\cdot 4^{t_i/9}}_{\#\text{ starfish}})$ sq. meters of coral are eaten

$\left(\text{sq meters}\cdot\frac{\text{yr}}{\text{starfish}\cdot\text{yr}}\right)$

taken: $\int_0^{30} 5\cdot 40e^{\frac{\ln 4}{9}t}\,dt = 200\int_0^{30} e^{\frac{\ln 4}{9}t}\,dt = 200\,\frac{9}{\ln 4}\,e^{\frac{\ln 4}{9}t}\Big|_0^{30}$

$= \frac{1800}{\ln 4}\left(e^{\frac{30}{9}\ln 4} - 1\right) = \frac{900}{\ln 2}\left(4^{10/3} - 1\right)$ (since $\ln 4 = 2\ln 2$)

ally - since the # of starfish is not constant throughout this time period, we must integrate.

could start by looking at the rate of change of coral: $\frac{dC}{dt} = 5S \Rightarrow \int dC = \int 5S\,dt$

o you've got some integral as above.

$(\ldots 0)\cdot 5$ is wrong. You're looking at the # of starfish in 1990 - how much they would eat in 1 year.

2. (9 points) Consider the system of differential equations

$$\frac{dx}{dt} = -y$$

$$\frac{dy}{dt} = x + xy = x(1+y)$$

(a) For what pairs (x, y) is the system at equilibrium?

$\frac{dx}{dt} = 0$ when $y = 0$

$\frac{dy}{dt} = 0$ when $x = 0$ or $y = -1$

Both are zero when $x = 0$

(i.e.,

(b) Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{x + xy}{-y}.$$

You need not solve explicitly for y or for x.

$$\frac{dy}{dx} = \frac{x(1+y)}{-y}$$

$$\int \frac{-y}{1+y}\,dy = \int x\,dx$$

$$\int \left[-1 + \frac{1}{1+y}\right]dy = \int x\,dx$$

$$-y + \ln|1+y| = \frac{x^2}{2} + C$$

(c) Sketch the solution curves in the xy plane "near" the point $(0,0)$. observe that near zero, $\frac{dy}{dt} \approx x$.) Please include arrows on your trajectories. attention to the 1^{st} quadrant.)

$\frac{dx}{dt} = -y$

$\frac{dy}{dt} \approx x$

$\frac{dy}{dx} \approx -\frac{x}{y} \rightarrow y\,dy = -x\,dx \rightarrow \frac{y^2}{2}$

CIRI

(Note: $\frac{dx}{dt} < 0$ when $y > 0$.... That gives arrow directions)