Harvard University
Math 1b. Second Semester Calculus
Second Exam

Name Answer Key

November 24, 1998

Second Exam

- Do not open this exam booklet until you are directed to do so.
- You have 120 minutes to earn 93 points.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Do not put part of the answer to one problem on the back of the sheet for another problem.
- Do not spend too much time on any problem. Read them all through first and attack them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat. By the same token, be sure to justify your solutions, so we can follow your reasoning.
- Good luck!

Problem	Points	Grade
1	18	
2	15	
3	10	
4	10	
5	15	
6	15	
7	10	
Total	93	

Please circle your section:

MWF 10:00 Brian Conrad	MWF 10:00 Andy Engelward	MWF 10:00 Eric Towne	MWF 11:00 Noam Elkies	MWF 11:00 Robert Pollack	MWF 12:00 Andy Engelward
TTh 10:00 Tomas Klenke	TTh 10:00 Joel Rosenberg	TTh 10:00 Eric Towne	TTh 11:30 Heather Russell	TTh 11:30 Nadia Lapusta	

1. (18 pts) Evaluate the following integrals.

a. $\displaystyle\int_0^2 \frac{3x}{(1+x^2)^2}\,dx$

sub $u = 1+x^2$

$du = 2x\,dx$ or $\frac{1}{2}du = x\,dx$

Then $\displaystyle\int \frac{3x}{(1+x^2)^2}dx = \int \frac{3\cdot\frac{1}{2}du}{u^2} = \frac{3}{2}\left(-u^{-1}\right) + C$

$= -\dfrac{3}{2(1+x^2)}$

so $\displaystyle\int_0^2 \frac{3x}{(1+x^2)^2}dx = -\frac{3}{2(1+x^2)}\Big|_0^2$

$= -\dfrac{3}{2\cdot 5} - \left(-\dfrac{3}{2\cdot 1}\right) = -\dfrac{3}{10} + \dfrac{3}{2} = \dfrac{12}{10} = \dfrac{6}{5}$

b. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{\sin t}}\,dt$

Note $\sin t = 0$ when $t=0$, so improper integral

$\displaystyle= \lim_{M\to 0^+}\int_M^{\pi/2}\frac{\cos t}{\sqrt{\sin t}}dt$. Now to evalute $\displaystyle\int \frac{\cos t}{\sqrt{\sin t}}dt$

sub $u = \sin t$, $du = \cos t$ Then $\displaystyle\int\frac{\cos t}{\sqrt{\sin t}}dt = \int u^{-1/2}du$

$= 2u^{1/2}$

$= 2\sqrt{\sin t}$

so original integral is

$\displaystyle\lim_{M\to 0^+}\left(2\sqrt{\sin t}\,\Big|_M^{\pi/2}\right) = 2$

c. $\displaystyle\int e^{\sqrt{x}}\,dx$

sub $w = \sqrt{x}$

$dw = \frac{1}{2}x^{-1/2}dx$, or $2\sqrt{x}\,du = dx$,

so $2w\,dw = dx$

Then $\displaystyle\int e^{\sqrt{x}}dx = 2\int we^w dw$. Now integrate by parts:

$u = w \quad dv = e^w dw$

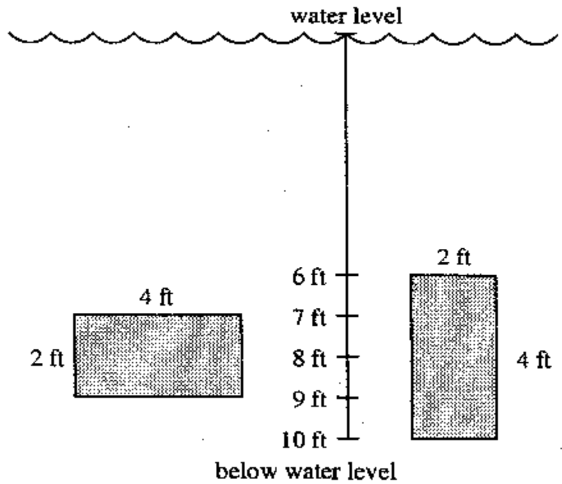
$du = dw \quad v = e^w$

$\hookrightarrow = 2\left[we^w - \int e^w dw\right]$

$= 2we^w - 2e^w + C$

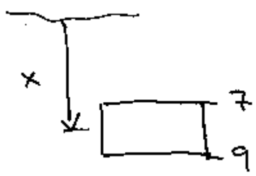
$= 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

2. (15 pts) The owner of a sight-seeing tour boat is going to install a 2 foot by 4 foot glass window under the water level on her boat so that people can look at the fish as the boat moves. She is considering orienting the window in one of the two configurations shown.



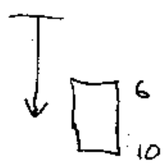
Calculate the fluid force on the window in each orientation and show whether one has a lower total fluid force on it. (Assume the sea water's density is $\rho = 64\frac{\text{lb}}{\text{ft}^3}$).

Left window:



depth = x feet
width = 4 ft.

$$\text{Fluid force} = \int_7^9 x \cdot \rho \cdot 4\,dx$$
$$= 4 \cdot 64 \cdot \left(\frac{x^2}{2} \Big|_7^9 \right)$$
$$= 4 \cdot 64 \cdot \left(\frac{81}{2} - \frac{49}{2} \right) = 4 \cdot 64 \cdot 16$$
$$= 64^2 \text{ lb.s}$$



Right window:
depth x feet
width = 2 feet

$$\text{Fluid force} = \int_6^{10} x \cdot \rho \cdot 2\,dx$$
$$= 2 \cdot 64 \cdot \left(\frac{x^2}{2} \Big|_6^{10} \right)$$
$$= 2 \cdot 64 \cdot \left(\frac{100}{2} - \frac{36}{2} \right)$$
$$= 2 \cdot 64 \cdot 32 = 64^2 \text{ lbs}$$

$\Longrightarrow$ same force on both

3. (10 pts) Evaluate the following integral.

$$\int \frac{7x+1}{(2x-1)(x+1)}\,dx$$

Partial fractions: $\dfrac{7x+1}{(2x-1)(x+1)} = \dfrac{A}{2x-1} + \dfrac{B}{x+1}$

so $A(x+1) + B(2x-1) = 7x+1$

sub $x=-1 \Rightarrow B\cdot(-2-1) = -7+1 = -6$

$B = 2$

sub $x = \frac{1}{2}$ $\quad A\left(\frac{3}{2}\right) = \frac{7}{2}+1 = \frac{9}{2}$

so $A = 3$

so integral is $\displaystyle\int \frac{3}{2x-1}\,dx + \int \frac{2}{x+1}\,dx$

$= \frac{3}{2}\ln\left|x-\frac{1}{2}\right| + 2\ln|x+1| + C$

4. (10 pts) Find the area of the surface generated by revolving the curve given by the equation $y = \frac{x^4}{4} + \frac{1}{8x^2}$ on the interval $[1, 2]$ about the ~~y-axis~~.
x-axis

so $\frac{dy}{dx} = x^3 - \frac{1}{4}x^{-3}$ and $1 + \left(\frac{dy}{dx}\right)^2$

$$= 1 + (x^3 - \tfrac{1}{4}x^{-3})^2$$
$$= 1 + x^6 - \tfrac{1}{2} + \tfrac{1}{16}x^{-6}$$
$$= (x^3 + \tfrac{1}{4}x^{-3})^2$$

so area $= \int_1^2 2\pi\left(\frac{x^4}{4} + \frac{1}{8x^2}\right)\sqrt{(x^3 + \tfrac{1}{4}x^{-3})^2}\,dx$

$$= \int_1^2 2\pi\left(\frac{x^4}{4} + \frac{1}{8x^2}\right)(x^3 + \tfrac{1}{4}x^{-3})\,dx$$

$$= \int_1^2 2\pi\left(\frac{x^7}{4} + \frac{x}{8} + \frac{x}{16} + \frac{1}{32x^5}\right)dx$$

$$= 2\pi\left[\frac{x^8}{32} + \frac{3}{32}x^2 - \frac{1}{128x^4}\right]\Bigg|_1^2$$

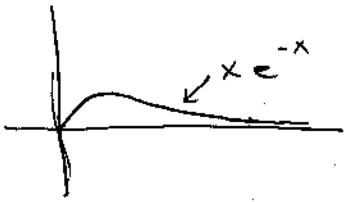
$$= 2\pi\left(\left[\frac{2^8}{32} + \frac{12}{32} - \frac{1}{2048}\right] - \left[\frac{1}{32} + \frac{3}{32} - \frac{1}{128}\right]\right)$$

$$= 2\pi\left(\frac{264}{32} + \frac{1}{128} - \frac{1}{2048}\right)$$ stopping here is fine

5. (15 pts) Let R be the region in the plane consisting of points whose (x, y) coordinates satisfy $x \geq 0$ and $0 \leq y \leq xe^{-x}$.

a. What is the area of R?

b. What is the volume of the solid obtained by rotating R about the x-axis?



a) R's area $= \underbrace{\int_0^\infty xe^{-x}\,dx}_{\text{improper!}} = \lim_{M\to\infty}\int_0^M xe^{-x}\,dx$

solve $\int xe^{-x}\,dx$ by parts $\quad u = x \quad dv = e^{-x}\,dx$, $\quad du = dx \quad v = -e^{-x}$

$= -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x}$

Then area $= \lim_{M\to\infty}\left(-xe^{-x} - e^{-x}\right)\Big|_0^M = \lim_{M\to\infty}\left[\left(-\frac{M}{e^M} - e^{-M}\right) - (-1)\right]$

$\lim_{M\to\infty} \frac{-M}{e^M} = \lim_{M\to\infty}\frac{-1}{e^M} = 0$, so $= 0 - 0 + 1 = 1$

b) Volume is $\int_0^\infty \pi\left(xe^{-x}\right)^2 dx = \int_0^\infty \pi x^2 e^{-2x}\,dx = \lim_{M\to\infty}\int_0^M \pi x^2 e^{-2x}\,dx$

$\pi\int x^2 e^{-2x}\,dx$: $\quad u = x^2 \quad dv = e^{-2x}\,dx$, $\quad du = 2x\,dx \quad v = -\frac{e^{-2x}}{2}$ $\quad$ integral $= \pi\left[-\frac{x^2e^{-2x}}{2} + \int xe^{-2x}\,dx\right]$

again by parts $\quad u = x \quad dv = e^{-2x}\,dx$, $\quad du = dx \quad v = -\frac{e^{-2x}}{2}$ $\quad$ integral then is equal to

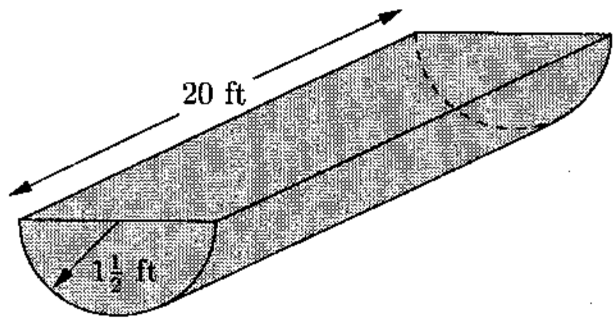
$$\pi\left[-\frac{x^2e^{-2x}}{2} + \left[-\frac{xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\,dx\right]\right]$$

$$= \pi\left[-\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4}\right]$$

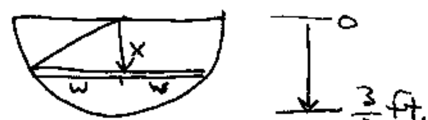
so volume $= \lim_{M\to\infty}\left[\pi\left(\underbrace{\frac{-M^2}{2e^{2M}}}_{0} - \underbrace{\frac{M}{2e^{2M}}}_{0} - \underbrace{\frac{1}{4e^{2M}}}_{0}\right) - \pi\left(-\frac{1}{4}\right)\right] = \frac{\pi}{4}$

by l'hopital

6. (15 pts) A wading pool has the shape of a half-cylinder of radius $1\frac{1}{2}$ feet and length 20 feet. That is, the surface is a 20ft $\times$ 3ft rectangle, and the pool has maximal depth $1\frac{1}{2}$ feet. The pool is full of water. Set up, but *do not evaluate,* an integral which computes the amount of work it would take to pump all the water to the level of the top of the ~~tank~~ pool. Use $\rho = 62.4\frac{\text{lb}}{\text{ft}^3}$ for the density of the water.



Note



if x measures depth then $w^2+x^2=\left(\frac{3}{2}\right)^2$, so width $=2w$
$= 2\sqrt{\frac{9}{4}-x^2}$

so work will be

$$\int_0^{3/2} \underset{\text{depth}}{x} \cdot \underset{\text{density}}{\rho} \cdot \underbrace{20 \cdot 2\sqrt{\frac{9}{4}-x^2}}_{\text{cross-section area}}\, dx$$

$$= (62.4)(40)\int_0^{3/2} x\sqrt{\frac{9}{4}-x^2}\, dx$$

7. (10 pts) If we rotate the curve given by $y = \sqrt{x(x-1)(x+1)}$ on the interval from $x = -1$ to $x = 0$ around the x-axis, what is the volume of the egg-shape we obtain?

volume is $\displaystyle\int_{-1}^{0} \pi \left(\sqrt{x(x-1)(x+1)}\right)^2 dx$

$$= \int_{-1}^{0} \pi \left(x(x-1)(x+1)\right) dx$$

$$= \pi \int_{-1}^{0} \left(x^3 - x\right) dx \quad = \left(\pi\frac{x^4}{4} - \pi\frac{x^2}{2}\right)\Bigg|_{-1}^{0}$$

$$= 0 - \left(\frac{\pi}{4} - \frac{\pi}{2}\right)$$

$$= \frac{\pi}{4}$$