Simpson's Rule as the Weighted average of the Midpoint and Trapezoidal Rules

To approximate $\int_a^b f(x)\,dx$, let's partition the interval $[a,b]$ into 8 equal pieces, each of length $\Delta x = \frac{b-a}{n}$. Let $x_i = a + \Delta x$ and let $y_i = f(x_i)$.

Think about using the trapezoidal and midpoint rules on intervals of length 2 Δx. In other words, although in the text it will that what we are about to do is Simpson's rule with $n = 8$, we're getting it by using $(T_4 + 2M_4)/3$.

The trapezoidal rule is the average of left and right hand sums:

left sum:

$$(y_0 + y_2 + y_4 + y_6) \cdot 2\Delta x$$

right sum:

$$(y_2 + y_4 + y_6 + y_8) \cdot 2\Delta x$$

Therefore, the Trapezoidal Rule is:

$$(y_0 + 2y_2 + 2y_4 + 2y_6 + y_8)\frac{2\Delta x}{2}$$

Simplifying, we get the following for the Trapezoidal Rule:

$$(y_0 + 2y_2 + 2y_4 + 2y_6 + y_8)\Delta x$$

Midpoint Sum:

$$(y_1 + y_3 + y_5 + y_7)2\Delta x$$

2 Midpoint Sum:

$$2(y_1 + y_3 + y_5 + y_7)2\Delta x$$

equivalently

$$(4y_1 + 4y_3 + 4y_5 + 4y_7)\Delta x$$

(Trap + 2 Mid)/3 =

$$(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + 4y_7 + y_8)\frac{\Delta x}{3}$$

More generally, Simpson's Rule for n subdivision, n even, gives:

$$(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + \dots 4y_{n-1} + y_n)\frac{\Delta x}{3}$$