

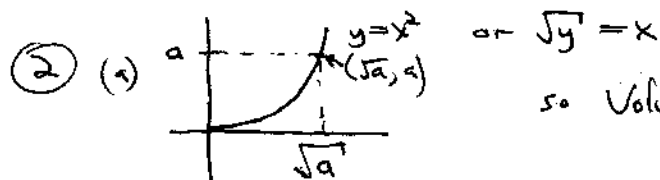
Answers for May 21, 1999 Math 1B Final Exam

① (a) Diverges: use divergence test $\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n}+10} = 1 \neq 0$

(b) Converges: use ratio test $\lim_{k \rightarrow \infty} \frac{(k+1)+2}{(k+1)!} \cdot \frac{k!}{k+2} = \lim_{k \rightarrow \infty} \frac{k+3}{(k+1)(k+2)}$
 $= \lim_{k \rightarrow \infty} \frac{k+3}{k^2+3k+2}$ which by L'Hopital $= \lim_{k \rightarrow \infty} \frac{1}{2k+3} = 0 < 1$

(c) Converges: $= \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ which is a geometric series with ratio $\frac{1}{e} < 1$

(d) $\sum_{n=1}^{\infty} \frac{\pi^2 n^3 - 15n}{n^5} = \sum_{n=1}^{\infty} \frac{\pi^2}{n^2} - \sum_{n=1}^{\infty} \frac{15}{n^4} = \pi^2 \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) - 15 \left(\sum_{n=1}^{\infty} \frac{1}{n^4} \right)$
 $= \pi^2 \left(\frac{\pi^2}{6} \right) - 15 \left(\frac{\pi^4}{90} \right)$
 $= \frac{\pi^4}{6} - \frac{\pi^4}{6} = 0$



so Volume $= \int_0^a \pi (\sqrt{y})^2 dy = \int_0^a \pi y dy$
 $= \pi \frac{y^2}{2} \Big|_0^a = \frac{\pi a^2}{2}$

(b) Surface Area $= \int_0^a 2\pi f(y) \sqrt{1+(f'(y))^2} dy$

here $f(y) = y^{1/2}$ $f'(y) = \frac{1}{2} y^{-1/2}$ $(f'(y))^2 = \frac{1}{4y}$

so surface area $= \int_0^a 2\pi (y^{1/2}) \sqrt{1 + \frac{1}{4y}} dy = \int_0^a 2\pi \sqrt{y + \frac{1}{4}} dy$

sub $u = y + \frac{1}{4}$ $du = dy$ surface area $= \int_{y=0}^{y=a} 2\pi \sqrt{u} du$

$= 2\pi \left(\frac{2}{3} u^{3/2} \right) \Big|_{y=0}^{y=a} = \frac{4\pi}{3} \left(\sqrt{y + \frac{1}{4}} \right)^3 \Big|_0^a$

$= \frac{4\pi}{3} \left(\left(a + \frac{1}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2} \right)$

$$\textcircled{3} \text{ (a) } y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots \quad a_0 = 1 \text{ since } y(0) = 1$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \dots \quad a_1 = 2 \text{ since } y'(0) = 2$$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots$$

$$-3y' = -3a_1 - 6a_2 x - 9a_3 x^2 - 12a_4 x^3 - 15a_5 x^4 - \dots$$

$$+2y = 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + 2a_4 x^4 + 2a_5 x^5 + \dots$$

$$\text{so } 2a_2 - 3a_1 + 2a_0 = 0 \quad \text{or } 2a_2 - 3 \cdot 2 + 2 \cdot 1 = 0 \Rightarrow a_2 = 2$$

$$6a_3 - 6a_2 + 2a_1 = 0 \quad \text{or } 6a_3 - 12 + 4 = 0 \Rightarrow a_3 = \frac{4}{3}$$

$$12a_4 - 9a_3 + 2a_2 = 0 \quad \text{or } 12a_4 - 12 + 4 = 0 \Rightarrow a_4 = \frac{2}{3}$$

$$\text{so } y = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \dots$$

$$\text{(b) this is } y = e^{2x}$$

$$\textcircled{4} \text{ (a) sub } u = \ln x \quad du = \frac{1}{x} dx \quad \int \frac{dx}{x} (\ln x)^{-1} = \int u^{-1} du$$

$$= \ln |u| + c = \ln |\ln x| + c$$

$$\text{(b) } \frac{d}{dx} (\ln |\ln x| + c) = \frac{1}{\ln x} \cdot \frac{d}{dx} (\ln x) = \frac{1}{(\ln x) \cdot x}$$

$$\text{(c) sub } u = \sin x \quad du = \cos x dx$$

$$= \int_{x=0}^{x=\pi/2} u du = \frac{u^2}{2} \Big|_{x=0}^{x=\pi/2} = \frac{\sin^2 x}{2} \Big|_0^{\pi/2} = \frac{1}{2}$$

$$\text{(d) split into } \lim_{a \rightarrow 0^-} \int_{-1}^a \frac{dx}{x^4} + \lim_{b \rightarrow 0^+} \int_b^1 \frac{dx}{x^4}$$

$$\text{first compute } \lim_{a \rightarrow 0^-} \left(\int_{-1}^a \frac{dx}{x^4} \right) = \lim_{a \rightarrow 0^-} \left(-\frac{x^{-3}}{3} \Big|_{-1}^a \right) = \lim_{a \rightarrow 0^-} \left(\frac{-1}{3a^3} - \frac{1}{3} \right)$$

so whole integral diverges

(don't need to check second part of integral now)

$$\textcircled{5} \text{ (a) } \frac{dy}{dx} + (-1)y = e^x \quad \text{Integrating factor } e^{\int -1 dx} = e^{-x}$$

$$\text{so } e^{-x} y = \int e^x \cdot e^{-x} dx = \int 1 dx = x + c, \quad \text{so } y = x e^x + c e^x$$

(5) (b) if $y = xe^x + Ce^x$ then $\frac{dy}{dx} = xe^x + e^x + Ce^x$

so $\frac{dy}{dx} - y = (xe^x + e^x + Ce^x) - (xe^x + Ce^x) = e^x \quad \checkmark$

(6) (a) Ratio Test: $\lim_{k \rightarrow \infty} \left| \frac{3^{k+1} \cdot x^{k+1}}{(k+1)^{1/2}} \cdot \frac{k^{1/2}}{3^k \cdot x^k} \right| = \lim_{k \rightarrow \infty} \left| 3x \frac{k^{1/2}}{(k+1)^{1/2}} \right| = |3x|$

so $|3x| < 1 \Rightarrow -\frac{1}{3} < x < \frac{1}{3}$

check endpoints: when $x = -\frac{1}{3}$ $\sum_{k=1}^{\infty} \frac{3^k (-\frac{1}{3})^k}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$

converges by Alt. Series test

when $x = \frac{1}{3}$

get $\sum_{k=1}^{\infty} \frac{3^k (\frac{1}{3})^k}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$

diverges as this is a p-series with $p = \frac{1}{2} < 1$

so Interval of Conv. is $-\frac{1}{3} < x < \frac{1}{3}$

(Radius of conv. is $\frac{1}{3}$)

(b) $\lim_{k \rightarrow \infty} \left| \frac{(k+1)^2 (x-1)^{k+1}}{2^{k+1}} \cdot \frac{2^k}{k^2 (x-1)^k} \right| = \lim_{k \rightarrow \infty} \left| \frac{k^2 + 2k + 1}{k^2} \cdot \frac{|x-1|}{2} \right|$

$= \frac{|x-1|}{2} < 1$

$\Rightarrow -1 < \frac{x-1}{2} < 1$

or $-2 < x-1 < 2$

or $-1 < x < 3$

check endpoints:

for $x = -1$

$\sum_{k=1}^{\infty} \frac{k^2 (-2)^k}{2^k} = \sum_{k=1}^{\infty} (-1)^k \cdot k^2$

here $\lim_{k \rightarrow \infty} ((-1)^k \cdot k^2) \neq 0$ so diverges

for $x = 3$

$\sum_{k=1}^{\infty} \frac{k^2 \cdot 2^k}{2^k} = \sum_{k=1}^{\infty} k^2$ also diverges

$\lim_{k \rightarrow \infty} k^2 \neq 0$

so Interval of Convergence is $-1 < x < 3$

and Radius of convergence is 2

(6) (c)
$$\lim_{k \rightarrow \infty} \left| \frac{5^{k+1} \cdot (x - \frac{1}{2})^{2k+2}}{(k+3)!} \cdot \frac{(k+2)!}{5^k (x - \frac{1}{2})^{2k}} \right|$$

$$= \lim_{k \rightarrow \infty} \left| \frac{5 (x - \frac{1}{2})^2}{k+3} \right| = 0$$

so converges for all x
radius of conv. is ∞

(7) (a) Rate in - Rate out
Rate in = $2 \frac{\text{gal}}{\text{min}} \cdot \frac{1}{100} \frac{\text{lb}}{\text{gal}} = \frac{1}{50} \frac{\text{lb}}{\text{min}}$
Rate out = $\frac{C(t)}{80} \cdot \frac{2 \text{gal}}{\text{min}} = \frac{C}{40} \frac{\text{lb}}{\text{min}}$

so diff. eqn. is $\frac{dC}{dt} = \frac{1}{50} - \frac{C}{40}$ with $C(0) = 2$

(b) $\frac{dC}{dt} + \frac{1}{40} C = \frac{1}{50}$ Integrating factor is $e^{\int \frac{1}{40} dt} = e^{t/40}$

so $C e^{t/40} = \frac{1}{50} \int e^{t/40} dt = \frac{40}{50} (e^{t/40}) + k$
(a constant)

or $C(t) = e^{-t/40} \left(\frac{40}{50} e^{t/40} + k \right)$
 $= \frac{4}{5} + k e^{-t/40}$

since $C(0) = 2$, then $2 = \frac{4}{5} + k e^0 = \frac{4}{5} + k$
so $k = \frac{6}{5}$

Thus $C(t) = \frac{4}{5} + \frac{6}{5} e^{-t/40}$

(c) $\lim_{t \rightarrow \infty} \left(\frac{4}{5} + \frac{6}{5} e^{-t/40} \right) = \frac{4}{5}$, so the amount of chlorine approaches $\frac{4}{5}$ lb over time.

(8) (a) $f(x) = (1+x)^{1/4}$ $f^{(4)}(x) = \frac{-231}{256} (1+x)^{-15/4}$

$f'(x) = \frac{1}{4} (1+x)^{-3/4}$ so $f(0) = 1$ $f'''(0) = \frac{21}{64}$

$f''(x) = \frac{-3}{16} (1+x)^{-7/4}$ $f'(0) = \frac{1}{4}$ $f^{(4)}(0) = \frac{-231}{256}$

$f'''(x) = \frac{21}{64} (1+x)^{-11/4}$ $f''(0) = -\frac{3}{16}$

(8) (a) continued so $f(x) = 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{21}{64 \cdot 3!}x^3 - \frac{231}{256 \cdot 4!}x^4 + \dots$

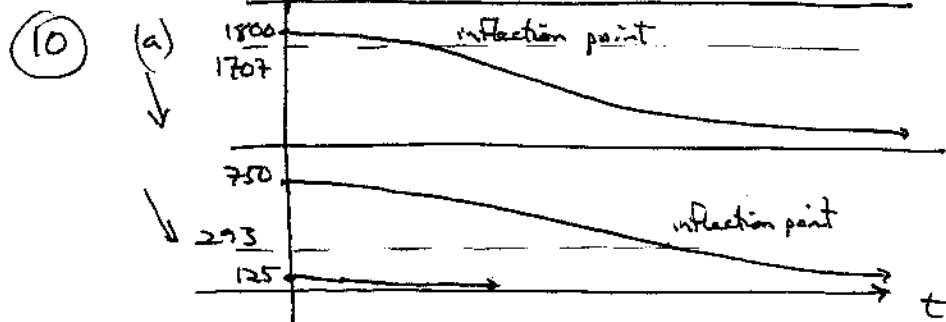
(b) $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$

so $\cos(2x) = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{4!} - \dots$
 $= 1 - \frac{4x^2}{2} + \frac{16x^4}{24} - \dots$

so $\sin^2 x = \frac{1}{2} - \frac{1}{2} \left(1 - 2x^2 + \frac{2}{3}x^4 - \dots \right)$
 $= \frac{1}{2} - \frac{1}{2} + x^2 - \frac{1}{3}x^4 = x^2 - \frac{x^4}{3} + \dots$

(9) (a) $\text{Work} = \int_{5000}^{6000} \frac{k}{r^2} dr = \left(-\frac{k}{r} \right) \Big|_{5000}^{6000} = -\frac{k}{6000} + \frac{k}{5000}$
 $= \frac{k}{30,000}$

(b) $\text{Work} = \int_{6000}^{\infty} \frac{k}{r^2} dr = \lim_{a \rightarrow \infty} \int_{6000}^a \frac{k}{r^2} dr$
 $= \lim_{a \rightarrow \infty} \left(-\frac{k}{r} \Big|_{6000}^a \right) = \lim_{a \rightarrow \infty} \left(\frac{k}{6000} - \frac{k}{a} \right)$
 $= \frac{k}{6000}$



(b) so for any $x(0) > 2000$ the spaceship will escape