Harvard University
Math 1b. Second Semester Calculus
Exam 2

Name ______________________

April 20, 1999

Exam 2

- Do not open this exam booklet until you are directed to do so.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Do not put part of the answer to one problem on the back of the sheet for another problem.
- Do not spend too much time on any one problem. Read them all through first and work on them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat. By the same token, be sure to justify your solutions (unless you are explicitly told otherwise), so we can follow your reasoning.
- Good luck!

Problem	Points	Grade
1	10	
2	10	
3	12	
4	8	
5	8	
6	13	
7	15	
8	14	
9	10	
Total	100	

Please circle your section:

MWF 10:00	MWF 11:00	MWF 12:00	TTh 10:00	TTh 11:30
Brian Conrad	Grisha Mikhalkin	Cathy O'Neil	Andy Engelward	Andy Engelward

1. (10 pts) For each of the following improper integrals, determine whether it converges or diverges, and compute the value of the integral in the convergent cases. Be sure to justify your answers.

a. (2 pts)

$$\int_0^\infty \cos(x)dx$$

$$= \lim_{b\to+\infty}\left(\sin x\Big|_0^b\right) = \underbrace{\lim_{b\to+\infty}\sin(b)}_{} - \lim_{b\to+\infty}\cancel{\sin(0)}^{\to 0}$$

limit does not exist because sin(x) oscillates between -1 & 1.

$\boxed{\text{Diverges}}$

b. (3 pts)

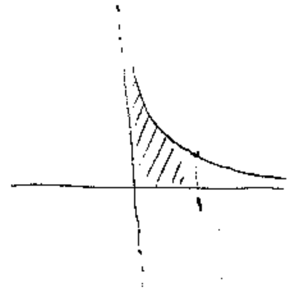
$$\int_1^\infty \frac{dx}{\sqrt{x}}$$

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to+\infty}\left(2\sqrt{x}\Big|_1^b\right) = \underbrace{\lim_{b\to+\infty}2\sqrt{b}}_{=+\infty} - \lim_{b\to+\infty}2\sqrt{1}$$

$\boxed{\text{Diverges}}$

c. (3 pts)

$$\int_0^1 \frac{dx}{\sqrt{x}}$$



vertical asymptote at $x=0$.

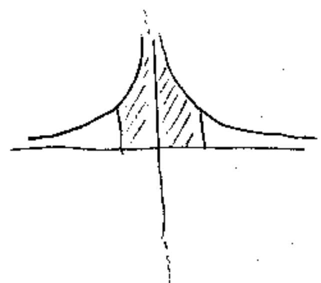
So compute $\lim_{a\to 0^+}\left(2\sqrt{x}\,\Big|_a^1\right)$

$= 2\sqrt{1} - \lim_{a\to 0^+} 2\sqrt{a} = 2 - 0 = \boxed{2}$

$\boxed{\text{Converges}}$

d. (2 pts)

$$\int_{-1}^1 \frac{dt}{t^{10}}$$



vertical asymptote at $x=0$

So compute $\underbrace{\int_{-1}^0 \frac{dt}{t^{10}}}_{\mathrm{I}}$ & $\underbrace{\int_0^1 \frac{dt}{t^{10}}}_{\mathrm{II}}$;

$\mathrm{I}: \int_{-1}^0 \frac{dt}{t^{10}} = \lim_{b\to 0^-}\left(\frac{-1}{9t^9}\Big|_{-1}^b\right) = \underbrace{\lim_{b\to 0^-}\frac{-1}{9b^9}}_{=+\infty} - \left(\frac{-1}{9(-1)^9}\right)$

So $\boxed{\text{diverges}}$

Note: You don't need to bother computing II.
Divergence of I is sufficient to conclude divergence

2. (10 pts) Show that

$$\int_0^\infty t^2 e^{-t} dt$$

converges and is equal to 2.

This is the product of a polynomial by an exponential, so try integration by parts. Since integrating e^{-t} doesn't complicate things, let's try:

$\int t^2 e^{-t} dt$: $\left(\begin{array}{ll} u = t^2 & v = -e^{-t} \\ du = 2t\,dt & dv = e^{-t}dt \end{array} \right)$

$\int t^2 e^{-t} dt = -t^2 e^{-t} - \int -2te^{-t} dt$ Integration by parts again:

$= -t^2 e^{-t} + 2\int te^{-t} dt$ $\left(\begin{array}{ll} u = t & v = -e^{-t} \\ du = dt & dv = e^{-t}dt \end{array} \right)$

$= -t^2 e^{-t} + 2\left[-te^{-t} - \int -e^{-t} dt \right]$

$= -t^2 e^{-t} - 2te^{-t} + 2\int e^{-t} dt$

$= \dfrac{-t^2}{e^t} - \dfrac{2t}{e^t} - \dfrac{2}{e^t} + C.$

Now we are ready to deal with the improper integral.

Note that $\lim_{t\to\infty} \dfrac{t^2}{e^t} = \lim_{t\to\infty} \dfrac{2t}{e^t} = \lim_{t\to\infty} \dfrac{2}{e^t} = 0$ by l'Hôpital's rule.

(or, you can appeal to the fact that this is a $\dfrac{\text{polynomial}}{\text{exponential}}$)

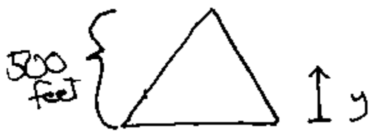
$\int_0^\infty t^2 e^{-t} dt = \lim_{b\to+\infty} \left[\left(\dfrac{-t^2}{e^t} - \dfrac{2t}{e^t} - \dfrac{2}{e^t} \right) \Big|_0^b \right]$

$= \lim_{b\to+\infty} \left(\dfrac{-b^2}{e^b} - \dfrac{2b}{e^b} - \dfrac{2}{e^b} \right) - (0 - 0 - 2) = \boxed{2}$

So $\boxed{\text{converges to } 2}$

3. (12 pts) A worker in ancient Egypt is asked to carry a bag of sand up to the top of the main pyramid at Giza. The sand weighs 100 lbs when he starts up the pyramid, but because the bag has a small hole in it, the sand leaks out at a constant rate. Exactly when the worker reaches the top of the pyramid, 500 feet up from where he started, the bag ends up completely empty. Assuming the worker walked up the pyramid at a constant rate and that the bag itself weighs 5 lbs, how much work (in units of ft-lbs) did he do in carrying up the bag? You must express your answer as an integer (e.g. 1000 instead of $500 + 2 \cdot 250$).

Let $F(y)$ = weight of the sand and the bag when the worker has reached height y above the base

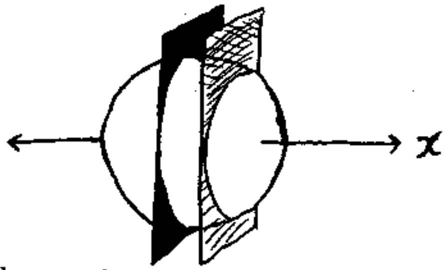


Since the sand is leaking out at a constant rate, then for each foot the worker goes up the bag loses $\frac{100 \text{ lbs}}{500 \text{ ft}}$, or so it loses $\frac{1}{5}$ lb per foot.

$$\text{So } F(y) = 5 \text{ lbs} + 100 \text{ lbs} - \tfrac{1}{5} y \text{ lbs}$$
$$= \left(105 - \frac{y}{5}\right) \text{ lbs}$$

$$\text{Thus work} = \int_0^{500} \left(105 - \frac{y}{5}\right) dy$$
$$= \left(105y - \frac{y^2}{10}\right)\Bigg|_0^{500}$$
$$= 105 \cdot 500 - \frac{500^2}{10}$$
$$= 52{,}500 - \frac{250{,}000}{10} = \boxed{27{,}500 \text{ ft-lbs}}$$

4. (8 pts) Let S be the surface obtained by revolving the semi-circle $x^2+y^2=1, y \geq 0$ around the x-axis; S is the surface of a sphere with radius 1, centered at the origin. Two planes perpendicular to the x-axis cut through S as shown.



Suppose the planes cut the x-axis at $x=a$ and $x=b$ respectively, with $-1 \leq a < b \leq 1$.

a. (3 pts) Compute the volume of the region inside the sphere which lies between the two planes. Your answer should depend on a and b.

In general, $V = \int_a^b \pi [f(x)]^2 \, dx$.

Here, $f(x) = y = \sqrt{1-x^2}$.

$$\text{So } V = \int_a^b \pi \left[\sqrt{1-x^2}\right]^2 dx = \int_a^b \pi(1-x^2)\,dx = \left(\pi x - \frac{\pi x^3}{3}\right)\Big|_a^b$$

$$= \left(\pi b - \frac{\pi b^3}{3}\right) - \left(\pi a - \frac{\pi a^3}{3}\right)$$

$$= \boxed{\pi(b-a) + \frac{\pi}{3}(a^3-b^3)}$$

b. (1 pt) Check that when $a=-1$ and $b=1$ your above answer recovers the volume $\frac{4}{3}\pi$ of a sphere of radius 1.

when $a=-1$, $b=-1$, $V = \pi(1-(-1)) + \frac{\pi}{3}\left((-1)^3 - 1^3\right)$

$$= \pi + \pi - \frac{\pi}{3} - \frac{\pi}{3} = 2\pi - \frac{2\pi}{3} = \boxed{\frac{4\pi}{3}}$$

c. (4 pts) Show that the surface area of the part of S that lies between the two planes is $2\pi(b-a)$. Note the remarkable fact that this depends only on the distance $b-a$ between the two planes, not on the values of a and b separately.

In general, $S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$

Here, $f(x) = \sqrt{1-x^2}$, $f'(x) = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$

So $S = \int_a^b 2\pi\sqrt{1-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx$

$$= \int_a^b 2\pi\sqrt{(1-x^2)+(1-x^2)\frac{x^2}{(1-x^2)}}\,dx$$

$$= \int_a^b 2\pi\sqrt{1-x^2+x^2}\,dx$$

$$= \int_a^b 2\pi\,dx = \boxed{2\pi(b-a)}$$

5. (8 pts) Evaluate

$$\int \frac{\cos(x)}{4-\sin^2(x)}dx.$$

These functions look related, so let's try a substitution.

let $u = \sin x$, $du = \cos x\, dx$.

$$\int \frac{\cos x}{4-\sin^2 x}dx = \int \frac{du}{4-u^2} = \int \frac{du}{(2+u)(2-u)}$$

Partial fractions: $\frac{A}{2+u} + \frac{B}{2-u} = \frac{1}{(2+u)(2-u)}$

$A(2-u) + B(2+u) = 1.$

@ $u=2$: $0 + 4B = 1 \implies B = 1/4$

@ $u=-2$: $A \cdot 4 + 0 = 1 \implies A = 1/4$

So $\int \frac{du}{(2+u)(2-u)} = \int \frac{1}{4}\frac{du}{2+u} + \int \frac{1}{4}\frac{du}{2-u}$

$= \frac{1}{4}\ln|2+u| - \frac{1}{4}\ln|2-u| + C$

$= \boxed{\frac{1}{4}\ln|2+\sin x| - \frac{1}{4}\ln|2-\sin x| + C}$

If desired, this can be simplified to:

$= \ln|2+\sin x|^{1/4} - \ln|2-\sin x|^{1/4} + C$

$= \boxed{\ln\left|\frac{2+\sin x}{2-\sin x}\right|^{1/4} + C}$

6. (13 pts) Evaluate

$$\int \frac{x^2+1}{x \cdot (x^2-1)} dx.$$

$\int \frac{(x^2+1)\,dx}{x(x^2-1)}$ is not of form $\int \frac{f'(x)\,dx}{f(x)}$, but deg(numerator) < deg(denominator), so we can use partial fractions.

$= \int \frac{(x^2+1)\,dx}{x(x+1)(x-1)}$: $\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} = \frac{x^2+1}{x(x+1)(x-1)}$

$A(x^2-1) + Bx(x-1) + Cx(x+1) = x^2+1.$

@ $x=1$: $A \cdot 0 + B \cdot 0 + C \cdot 2 = 1+1 \Rightarrow \boxed{C=1}$

@ $x=0$: $A(-1) + B \cdot 0 + C \cdot 0 = 0+1 \Rightarrow \boxed{A=-1}$

@ $x=-1$: $A \cdot 0 + B(-1)(-2) + C \cdot 0 = 1+1 \Rightarrow \boxed{B=1}$

$= \int \frac{-dx}{x} + \int \frac{dx}{x+1} + \int \frac{dx}{x-1}$

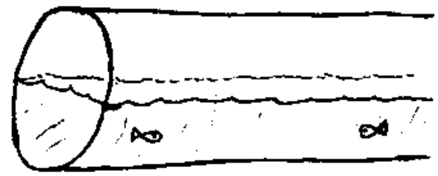
$= \boxed{-\ln|x| + \ln|x+1| + \ln|x-1| + C}$ can be simplified to:

$= -\ln|x| + \ln|x^2-1| + C$

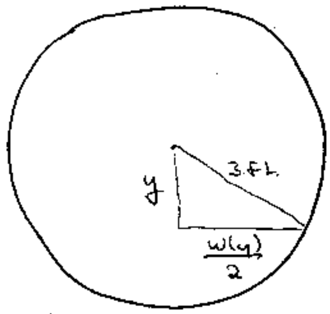
$= \ln\left|\frac{x^2-1}{x}\right| + C$

$= \boxed{\ln\left|x - \frac{1}{x}\right| + C}$

7. (15 pts) Kourtney has a large glass fish tank in the shape of a cylindrical pipe of diameter 6 feet, as shown below. Both circular ends of the tank are flat glass disks. The tank is half filled with water.



What is the force (in lbs) of the water against the circular end shown? The density of water is 62.4 lbs/ft^3.



$w(y)$ = width of tank at depth of y ft below central axis.

By Pythag., $\frac{w(y)}{2} = \sqrt{9-y^2}$

The force on the bottom half of the disk is given by:

$$F = \int_{y=0}^{y=3} \rho \cdot w(y) \cdot y \, dy = \int_0^3 62.4 \left(2\sqrt{9-y^2}\right) y \, dy$$

$$= 62.4(-1)\left(\frac{2}{3}\right)(9-y^2)^{3/2} \Big|_0^3$$

$$= (-20.8)(2)\left[(9-3^2)^{3/2} - 9^{3/2}\right]$$

$$= -41.6\left[0-27\right] = \boxed{27(41.6) \text{ lbs}}$$

8. (14 pts) Evaluate the following integrals (be sure to show all your work, not just the final answer).

a. (2 pts)

$$\int x(1-x)^{1999}dx$$

Substitution: $u=1-x \Rightarrow x=1-u$

$du=-dx$

$$\int x(1-x)^{1999}dx = \int u^{1999}(1-u)(-du)$$
$$=\int(u-1)(u^{1999})du$$
$$=\int\left(u^{2000}-u^{1999}\right)du$$
$$=\frac{u^{2001}}{2001}-\frac{u^{2000}}{2000}+C$$
$$=\boxed{\frac{(1-x)^{2001}}{2001}-\frac{(1-x)^{2000}}{2000}+C}$$

b. (3 pts)

$$\int x\sqrt{x+1}\sqrt{x-1}dx$$

$$=\int x\sqrt{x^2-1}\,dx$$ substitution $u=x^2-1$, $du=2x\,dx$

$$=\int\frac{\sqrt{u}}{2}du=\frac{1}{3}u^{3/2}+C$$
$$=\boxed{\frac{(x^2-1)^{3/2}}{3}+C}$$

c. (3 pts)

$$\int x(\ln(x))^2dx$$

By parts: $\int x(\ln(x))^2dx$ $\quad u=(\ln x)^2 \quad v=\frac{x^2}{2}$, $\quad du=\frac{2\ln x}{x}dx \quad dv=x\,dx$

$$=\frac{x^2}{2}(\ln x)^2-\int\frac{x^2}{2}\cdot\frac{2\ln x}{x}dx=\frac{x^2}{2}(\ln x)^2-\int x\ln x\,dx$$ $\quad u=\ln x \quad v=\frac{x^2}{2}$, $\quad du=\frac{dx}{x} \quad dv=x\,dx$

$$=\frac{x^2}{2}(\ln x)^2-\left[\frac{x^2}{2}\ln x-\int\frac{x^2}{2}\frac{dx}{x}\right]=\frac{x^2}{2}(\ln x)^2-\frac{x^2}{2}\ln x+\frac{x^2}{4}+C$$
$$=\boxed{\frac{x^2}{2}\left[(\ln x)^2-\ln x+\frac{1}{2}\right]+C}$$

d. (3 pts)

$$\int \frac{\ln(x^2)}{2x}dx$$

substitution
let $z = x^2$

$dz = 2x\,dx \Rightarrow \frac{dx}{2x} = \frac{dz}{4z}$

$\int \frac{\ln(x^2)dx}{2x} = \int \frac{\ln(z)dz}{4z}$

by parts: $u = \ln(z)$ $\quad v = \frac{1}{4}\ln|z|$

$du = \frac{dz}{z}$ $\quad dv = \frac{dz}{4z}$

$\int \frac{\ln(z)dz}{4z} = \frac{1}{4}\ln(z)\cdot\ln|z| - \int \frac{\ln(z)dz}{4z}$

$\int \frac{\ln(z)dz}{4z} = \frac{1}{8}\ln(z)\cdot\ln|z|$

So $\int \frac{\ln(x^2)dx}{2x} = \boxed{\frac{[\ln(x^2)]^2}{8} + C}$

Note: you can also start by noting that
$\frac{\ln(x^2)}{2x} = \frac{2\ln(|x|)}{2x} = \frac{\ln(|x|)}{x}$

e. (3 pts)

$$\int \frac{\sin(\sqrt{t})}{\sqrt{t}}dt$$

Substitution: $u = \sqrt{t}$

$du = \frac{1}{2\sqrt{t}}dt \Rightarrow \frac{dt}{\sqrt{t}} = 2\,du$

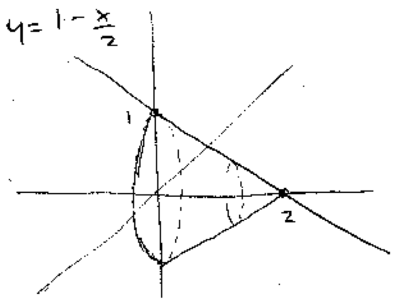
$\int \frac{\sin(\sqrt{t})dt}{\sqrt{t}} = \int 2\sin(u)du = -2\cos(u) + C$

$= \boxed{-2\cos(\sqrt{t}) + C}$

9. (10 pts) Let S be the surface of revolution obtained by rotating the graph of

$$y = 1 - \frac{x}{2},$$

from $x = 0$ to $x = 2$, around the x-axis. Find the surface area of S.



In general, $S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$

$f(x) = 1 - \frac{x}{2}$; $f'(x) = -\frac{1}{2}$

$$S = \int_0^2 2\pi\left(1 - \frac{x}{2}\right)\sqrt{1 + \left(-\frac{1}{2}\right)^2}\,dx$$

$$= 2\pi \int_0^2 \left(1 - \frac{x}{2}\right)\frac{\sqrt{5}}{2}\,dx$$

$$= \pi\sqrt{5}\left[\left(x - \frac{x^2}{4}\right)\Big|_0^2\right] = \pi\sqrt{5}\,(2-1) = \boxed{\pi\sqrt{5}}$$

Alternative Method: This surface is just a cone with radius 1 & height 2.

So $S = \pi\,(\text{radius})(\text{slant height})$

$$= \pi\,(1)\left(\sqrt{1+2^2}\right) = \boxed{\pi\sqrt{5}}$$