Solutions to the December 3rd Math 1b Exam

Problem 1.

(i) Answer c.

This is an autonomous differential equation; the rate of change of w is governed entirely by w itself.

It has equilibrium solutions of $w(t) = 2$ and $w(t) = -1$. This rules out (j) and (k).

For $w > 2$ the expression $(w-2)^2(w+1)$ is positive, so the solutions must be increasing. The choices now available are: (c), (d), (f), (g), and (i).

At $w = 0$ the expression $(w-2)^2(w+1)$ is positive, so the solutions must be increasing in the strip between $w(t) = 2$ and $w(t) = -1$. This leaves choices (c) and (i) only.

For $w < -1$ the expression $(w-2)^2(w+1)$ is negative, so the solutions must be decreasing. This leaves only choice (c).

(ii) Answer d.

In order to have a stable equilibrium at $P = 5$, it is first necessary that $P = 5$ is an equilibrium solution, i.e. that if $P = 5$ then $\frac{dP}{dt} = 0$. This leaves only three options: (b), (c), and (d).

In order to have a stable equilibrium at $P = 5$, for P a bit greater than 5 $\frac{dP}{dt}$ must be negative and for P a bit less than 5 $\frac{dP}{dt}$ must be positive. (Draw a picture of a stable equilibrium to see why this is true.)

Choice (b) has $\frac{dP}{dt} > 0$ for $P > 5$, so it is ruled out. Choice (c) has $\frac{dP}{dt} > 0$ for all $P \neq 5$, so it is ruled out. Check that choice (d) satisfies the condition above.

(iii) Answer c.

The equilibrium solutions do not help us distinguish between answers. All the choices are autonomous differential equations. The decision must be based on the sign of $\frac{dy}{dt}$.

We require the following:

For $y > 1$, $\frac{dy}{dt} > 0$ For $-2 < y$, $\frac{dy}{dt} > 0$ For $y < -2$, $\frac{dy}{dt} < 0$

(Notice that this problem is the complement to problem (i).)

Choice (c) is the only choice satisfying these requirements.

Problem 2.

(a) The graph of $f(x) = 1/x^2$ is symmetric about the $y-$axis, is always positive, and has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$.

The integrand of the $\displaystyle\int_{-2}^{2} \frac{dx}{x^2}$ is undefined and unbounded at $x = 0$. Therefore, we must look at

$$\int_{-2}^{0} \frac{dx}{x^2} + \int_{0}^{2} \frac{dx}{x^2}$$

and if both these integrals converge then the original integral converges to the sum. In fact, if $\int_0^2 \frac{dx}{x^2}$ converges the original integral is twice its value and if $\int_0^2 \frac{dx}{x^2}$ diverges the original integral does as well.

To determine whether of not $\int_0^2 \frac{dx}{x^2}$ converges we compute

$$\lim_{b\to 0^+} \int_b^2 \frac{dx}{x^2}.$$

$$\lim_{b\to 0^+} \int_b^2 \frac{dx}{x^2} = \lim_{b\to 0^+} \frac{-1}{x}\Big|_b^2 = \lim_{b\to 0^+} \frac{-1}{2} + \frac{1}{b} = \infty$$

Therefore, the original integral diverges.

(b) The answer is (iii), we don't have enough information to determine this.

There are many ways to reason this through. First of all, the behavior of the function on say $(-\infty, 1)$ is not really an issue. Any function we chose to think about on $(1, \infty)$ can be put together with another function tosatisfy the requirements on $(-\infty, \infty)$, so we'll just concern ourselves with the interval $(1, \infty)$ below.

One is to compute the integrals $\displaystyle\int_1^\infty \frac{dx}{x^2}$ and $\displaystyle\int_1^\infty \frac{dx}{x}$ and show that the former converges while the later diverges. (You computed both of these for homework when you were deriving results about $p-$ series for homework.) In both cases the integrand is positive and continuous on $(1, \infty)$ and tends towards 0 as $x \to \infty$.

Alternatively, you can think about the relationship between improper integrals of the form $\displaystyle\int_1^\infty f(x)\, dx$ and the corresponding series. We know that if the terms of a series tend towards 0 that is not enough to assure that the series converges. (Think about the harmonic series.) Similarly, it is not enough for the integrand of $\int_1^\infty f(x)\, dx$ to be tending towards zero in order for the integral to converge. Tending towards zero is necessary but not sufficient. $\int_1^\infty \frac{dx}{x}\, dx$ is an example of such an improper integral that diverges.

You can argue that the conditions on the integrand assure that eventually the function $\int_1^t f(x)\, dx$ is an increasing concave down function, but a function can be increasing and concave down and have a finite limit or it can be increasing and concave down and grow without bound.

Problem 3.

(a) We first solve this problem using "cylinders" (*i.e.*, slicing vertically). We slice as in figure 3a1. The ith slice has approximate volume

$$2\pi \cdot (2 + x_i) \cdot \sin(x_i) \cdot \Delta x.$$

So the approximate total volume is given by the Riemann sum

$$2\pi \sum_{i=1}^{n} (2 + x_i) \cdot \sin(x_i) \cdot \Delta x.$$

Taking the limit of this Riemann sum, we obtain that the actual volume is given by the integral

$$2\pi \int_0^{\pi/2} (2 + x) \cdot \sin(x)\, dx.$$

We now solve this problem using "washers" (*i.e.*, slicing horizontally). We slice as in figure 3a2. The ith slice has approximate volume

$$\pi \cdot \left[(2 + \pi/2)^2 - (2 + \arcsin(y_i))^2\right] \Delta y.$$

So the approximate total volume is given by the Riemann sum

$$\pi \sum_{i=1}^{n} \left[(2 + \pi/2)^2 - (2 + \arcsin(y_i))^2\right] \Delta y.$$

Taking the limit of this Riemann sum, we obtain that the actual volume is given by the integral

$$\pi \int_0^1 \left[(2 + \pi/2)^2 - (2 + \arcsin(y))^2\right] dy.$$

(b) We first solve this problem using "cylinders" (*i.e.*, slicing horizontally). We slice as in figure 3b1. The ith slice has approximate volume

$$2\pi \cdot (3 + y_i) \cdot (\pi/2 - \arcsin(y_i)) \cdot \Delta y.$$

So the approximate total volume is given by the Riemann sum

$$2\pi \sum_{i=1}^{n} (3 + y_i) \cdot (\pi/2 - \arcsin(y_i)) \cdot \Delta y.$$

Taking the limit of this Riemann sum, we obtain that the actual volume is given by the integral

$$2\pi \int_0^1 (3 + y) \cdot (\pi/2 - \arcsin(y))\, dy.$$

We now solve this problem using "washers" (*i.e.*, slicing vertically). We slice as in figure 3b2. The ith slice has approximate volume

$$\pi \cdot \left[3^2 - (3 - \sin(x_i))^2\right] \Delta x.$$

So the approximate total volume is given by the Riemann sum

$$\pi \sum_{i=1}^{n} \left[3^2 - (3 - \sin(x_i))^2\right] \Delta x.$$

Taking the limit of this Riemann sum, we obtain that the actual volume is given by the integral

$$\pi \int_0^{\pi/2} \left[3^2 - (3 - \sin(x))^2\right] dx.$$

Problem 4.

(a) Rate of change = rate of increase - rate of decrease, so $\frac{dP}{dt} = 0.1P - 10^{-7}P^3$.

(b) Set $\frac{dP}{dt} = 0$. So $0 = 0.1P(1 - 10^{-6}P^2$. Then $P = 0$ or $P^2 = 10^6$, i.e. $P = 10^3 = 1000$.
The equilibrium levels are $P = 0$ and $P = 1000$.

(c) Evaluate $\frac{dP}{dt}$ at $P = 100$. We get $10 - 10^{-7} \cdot (10^2)^3 = 10 - 10^{-7+6} = 10 - .1 = 9.9$
The rabbit population is increasing at a rate of 9.9 rabbits per year.

Problem 5. The helicopter rises at 1 meter/second, so it takes y seconds to rise y meters. Since fuel is burned at a rate of 1 kg/second, at a height of y meters the helicopter has burned y kg of fuel. The helicopter's mass at height y is thus $56000 - y$.
The force at height y is

$$F = mg = (56000 - y)g$$

and the total work done (in Joules) is

$$\int_{y=0}^{2000} F\, dy = \int_0^{2000} (56000 - y)g\, dy = g\left(56000y - \frac{y^2}{2}\right)\Bigg|_0^{2000}$$

$$= g\left(56000 \cdot 2000 - \frac{2000^2}{2}\right) = 110000000g$$

Problem 6.

$$\frac{dy}{dx} = x^2 e^{y-x^3} \quad (1)$$
$$e^{-y}dy = x^2 e^{-x^3} dx \quad (2)$$
$$\int e^{-y}dy = \int x^2 e^{-x^3}\, dx \quad (3)$$

We use the substitution $u = -x^3$ to solve the integral on the right. We have

$$\int x^2 e^{-x^3} dx = \int \frac{-1}{3} e^u du \quad (4)$$
$$= \frac{-1}{3} e^u + C \quad (5)$$
$$= \frac{-1}{3} e^{-x^3} + C \quad (6)$$

So we have:

$$-e^{-y} = \frac{-1}{3} e^{-x^3} + C \quad (7)$$
$$e^{-y} = \frac{1}{3} e^{-x^3} + C \quad (8)$$
$$-y = \ln \mid \frac{1}{3} e^{-x^3} + C \mid \quad (9)$$
$$y = -\ln \mid \frac{1}{3} e^{-x^3} + C \mid \quad (10)$$

Plug in the initial condition $y(0) = 0$ we have $0 = -\ln \mid \frac{1}{3} + C \mid$. Since $\ln(1) = 0$ we must have $C = \frac{2}{3}$.

$$y = -\ln \mid \frac{1}{3} e^{-x^3} + \frac{2}{3} \mid$$

Problem 7. The answer is (c).
The density varies with the height, so we slice this cylinder into rectangular slabs parallel to the ground. We partition the interval $[0, 100]$ into n equal subintervals, each of length Δz. Let $z_i = i \cdot \Delta z$. The thickness of each slab is Δz cm and the length is 3000 cm. The width of each slab is most simply found by looking head-on into the cylindrical pipe - i.e. by looking at the circle of radius 50. Look at the width of the ith slice, a slice at a height of z_i. Make a triangle whose sides are the radius (length 50), half the width - let's call it x_i - and $|50 - z_i|$. Use the Pythagorean Theorem to find x_i in terms of z_i.

$$x_i^2 + (50 - z_i)^2 = 50^2$$
$$x_i = \sqrt{50^2 - (50 - z_i)^2}$$

The width of the slice is twice x_i.
Therefore, the volume of each slice is approximately given by $2\sqrt{50^2 - (50 - z_i)^2} \cdot 3000\Delta z$.
To approximate the mass of a slice, multiply the volume by density:$\rho(z_i)2\sqrt{50^2 - (50 - z_i)^2} \cdot 3000\Delta z$.
The total mass is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} \rho(z_i)2\sqrt{50^2 - (50 - z_i)^2} \cdot 3000\Delta z = \int_0^{100} \rho(z)2\sqrt{50^2 - (50 - z)^2} \cdot 3000\, dz$$

Problem 8.

(a) We must find C such that

$$\int_{-\infty}^{\infty} \frac{C}{1+x^2}\,dx = 1.$$

$\frac{C}{1+x^2}$ is an even function, so we can find C such that $\int_0^\infty \frac{C}{1+x^2}\,dx = 1/2$.

$$\int_0^\infty \frac{C}{1+x^2}\,dx = \lim_{b\to\infty} \frac{C}{1+x^2}\,dx = \lim_{b\to\infty} C\arctan(b) - C\arctan(0) = C\pi/2$$

Therefore, $C = \pi$.

(b) Because of symmetry, the answer is 1/2. If you didn't consider symmetry, then you'd need to compute.

(c) The probability of X being larger than $\sqrt{3}$ is $1/2 - P(0 < X < \sqrt{3})$. $1/2 - 1/3 = 1/6$. The probability is 1/6.

Alternatively, you could compute $\int_{\sqrt{3}}^\infty \frac{\pi}{x^2+1}\,dx$.

Problem 9. The correct answer is that

$$R_{15} < M_{15} < I < T_{15} < L_{15}.$$

5 points were given for stating this ordering, regardless of how substantial its justification was.

In addition to that, 1 point was given for stating that the inequality $R_{15} < I < L_{15}$ is due to the fact that the function f is *decreasing*. (Indeed on any subinterval $[x_k, x_{k+1}]$, for all $x \in (x_k, x_{k+1})$, we have $f(x_k) > f(x) > f(x_{k+1})$, so $f(x_k)\Delta x_k > \int_{x_k}^{x_{k+1}} f(x)dx > f(x_{k+1})\Delta x_k$. But $f(x_k)\Delta x_k$ is just the component of L_{15} corresponding to the subinterval $[x_k, x_{k+1}]$ and $f(x_{k+1})\Delta x_k$ is the component of R_{15} corresponding to the subinterval $[x_k, x_{k+1}]$. Summing up all components we conclude that $R_{15} < I < L_{15}$. This full explanation was not expected on the exam.)

1 point was given for saying that $R_{15} < M_{15} < L_{15}$ because the function is decreasing, and for saying that $R_{15} < T_{15} < L_{15}$ because T_{15} is an average of the two.

1 point was given for stating that $M_{15} < I$ because the function is concave up. (An explanation of how exactly up-concavity implies $M_{15} < I$ was not expected.)

1 point was given for stating that $I < T_{15}$ due to the fact that the function is concave up. (Again, an explanation of how exactly $f''(x) > 0$ implies $I < T_{15}$ was not expected.)

In general, suggestive pictures helping to justify the inequality $R_{15} < M_{15} < I < T_{15} < L_{15}$ were awarded 2-3 points in addition to the default 5.

Getting one inequality wrong amounted to subtraction of one additional point.