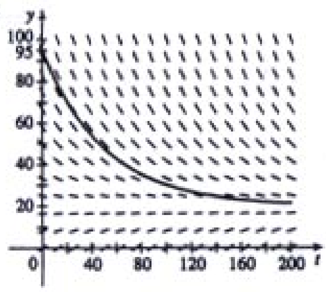
Solutions for PS 29

28. (a) From Exercise 7.1.14, we have $dy/dt = k(y - R)$. We are given that $R = 20$ °C and $dy/dt = -1$ °C/min when $y = 70$ °C. Thus, $-1 = k(70 - 20) \Rightarrow k = -\frac{1}{50}$ and the differential equation becomes $dy/dt = -\frac{1}{50}(y - 20)$.

(b)



The limiting value of the temperature is 20 °C; that is, the temperature of the room.

(c) From part (a), $dy/dt = -\frac{1}{50}(y - 20)$. With $t_0 = 0$, $y_0 = 95$, and $h = 2$ min, we get

$$y_1 = y_0 + hF(t_0, y_0) = 95 + 2\left[-\tfrac{1}{50}(95 - 20)\right] = 92$$
$$y_2 = y_1 + hF(t_1, y_1) = 92 + 2\left[-\tfrac{1}{50}(92 - 20)\right] = 89.12$$
$$y_3 = y_2 + hF(t_2, y_2) = 89.12 + 2\left[-\tfrac{1}{50}(89.12 - 20)\right] = 86.3552$$
$$y_4 = y_3 + hF(t_3, y_3) = 86.3552 + 2\left[-\tfrac{1}{50}(86.3552 - 20)\right] = 83.700992$$
$$y_5 = y_4 + hF(t_4, y_4) = 83.700992 + 2\left[-\tfrac{1}{50}(83.700992 - 20)\right] = 81.15295232$$

Thus, $y(10) \approx 81.15$ °C.

3. $yy' = x \Rightarrow y\frac{dy}{dx} = x \Rightarrow \int y\,dy = \int x\,dx \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1 \Rightarrow y^2 = x^2 + 2C_1 \Rightarrow$ $x^2 - y^2 = C$ (where $C = -2C_1$). This represents a family of hyperbolas.

4. $y' = xy \Rightarrow \int \frac{dy}{y} = \int x\,dx \quad [y \neq 0] \Rightarrow \ln|y| = \frac{x^2}{2} + C \Rightarrow |y| = e^C e^{x^2/2} \Rightarrow y = Ke^{x^2/2}$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

5. $\frac{dy}{dt} = \frac{te^t}{y\sqrt{1+y^2}} \Rightarrow y\sqrt{1+y^2}\,dy = te^t\,dt \Rightarrow \int y\sqrt{1+y^2}\,dy = \int te^t\,dt \Rightarrow$ $\frac{1}{3}(1+y^2)^{3/2} = te^t - e^t + C$ [where the first integral is evaluated by substitution and the second by parts] $\Rightarrow$ $1 + y^2 = [3(te^t - e^t + C)]^{2/3} \Rightarrow y = \pm\sqrt{[3(te^t - e^t + C)]^{2/3} - 1}$

6. $y' = \frac{xy}{2\ln y} \Rightarrow \frac{2\ln y}{y}\,dy = x\,dx \Rightarrow \int \frac{2\ln y}{y}\,dy = \int x\,dx \Rightarrow (\ln y)^2 = \frac{x^2}{2} + C \Rightarrow$ $\ln y = \pm\sqrt{x^2/2 + C} \Rightarrow y = e^{\pm\sqrt{x^2/2 + C}}$

9. $\frac{dy}{dx} = y^2 + 1$, $y(1) = 0$. $\int \frac{dy}{y^2 + 1} = \int dx \Rightarrow \tan^{-1} y = x + C$. $y = 0$ when $x = 1$, so $1 + C = \tan^{-1} 0 = 0 \Rightarrow C = -1$. Thus, $\tan^{-1} y = x - 1$ and $y = \tan(x - 1)$.

16. $\frac{dy}{dx} = \frac{y^2}{x^3}$, $y(1) = 1$. $\int \frac{dy}{y^2} = \int \frac{dx}{x^3} \Rightarrow -\frac{1}{y} = -\frac{1}{2x^2} + C$. $y(1) = 1 \Rightarrow -1 = -\frac{1}{2} + C \Rightarrow$ $C = -\frac{1}{2}$. So $\frac{1}{y} = \frac{1}{2x^2} + \frac{1}{2} = \frac{2 + 2x^2}{2 \cdot 2x^2} \Rightarrow y = \frac{2x^2}{x^2 + 1}$.

28. From Exercise 7.2.28, $\frac{dy}{dt} = -\frac{1}{50}(y - 20) \Leftrightarrow \int \frac{dy}{y - 20} = \int \left(-\frac{1}{50}\right) dt \Leftrightarrow \ln|y - 20| = -\frac{1}{50}t + C \Leftrightarrow$ $y - 20 = Ke^{-t/50} \Leftrightarrow y(t) = Ke^{-t/50} + 20$. $y(0) = 95 \Leftrightarrow 95 = K + 20 \Leftrightarrow K = 75 \Leftrightarrow$ $y(t) = 75e^{-t/50} + 20$.

33. (a) $\frac{dC}{dt} = r - kC \Rightarrow \frac{dC}{dt} = -(kC - r) \Rightarrow \int \frac{dC}{kC - r} = \int -dt \Rightarrow (1/k)\ln|kC - r| = -t + M_1$ $\Rightarrow \ln|kC - r| = -kt + M_2 \Rightarrow |kC - r| = e^{-kt + M_2} \Rightarrow kC - r = M_3 e^{-kt} \Rightarrow$ $kC = M_3 e^{-kt} + r \Rightarrow C(t) = M_4 e^{-kt} + r/k$. $C(0) = C_0 \Rightarrow C_0 = M_4 + r/k \Rightarrow$ $M_4 = C_0 - r/k \Rightarrow C(t) = (C_0 - r/k)e^{-kt} + r/k$.

(b) If $C_0 < r/k$, then $C_0 - r/k < 0$ and the formula for $C(t)$ shows that $C(t)$ increases and $\lim_{t\to\infty} C(t) = r/k$. As t increases, the formula for $C(t)$ shows how the role of C_0 steadily diminishes as that of r/k increases.