

## Solutions for PS 35

1. Solve the following differential equations for  $y(x)$ .

(a)  $y'' - 9y' = 0$

$$y(t) = C_1 e^{9t} + C_2$$

(b)  $y'' - 9y = 0$

$$y(t) = C_1 e^{3t} + C_2 e^{-3t}$$

(c)  $y'' + 9y = 0$

$$y(t) = C_1 e^{3ti} + C_2 e^{-3ti} = C_3 \cos(3t) + C_4 \sin(3t)$$

(d)  $y'' - 9 = 0$

$$y(t) = 4.5t^2 + C_1 t + C_2$$

(e)  $y'' - 2y' - y = 0$

$$y(t) = C_1 e^{(1+\sqrt{2})t} + C_2 e^{(1-\sqrt{2})t}$$

(f)  $y'' - 2y' + 2y = 0$   $y(t) = C_1 e^{(1+i)t} + C_2 e^{(1-i)t} = C_3 e^t \cos(t) + C_4 e^t \sin(t)$

2. Suppose that  $x'' + bx' + cx = 0$  is used to model the position of a block at the end of a vibrating spring.

(a) What can you say about the signs of  $b$  and  $c$ ? Explain.

$c$  must be positive because the roots of the characteristic equation need to be imaginary in order for the differential equation to describe oscillating motion (see question 4 from the Handout I), which requires a positive  $c$  value.

$b$  must be positive as well. There are a few ways of showing this. One is that, when the mass is at its equilibrium point,  $x = 0$ , so  $x'' = -bx'$ . Since the acceleration (at the equilibrium point, friction and drag are the only forces at play) is going in the opposite direction as the velocity, the value of  $b$  must be positive if the two sides of the equation  $x'' = -bx'$  are to have the same sign.

(b) As long as friction plays a role, we expect that regardless of the initial conditions  $\lim_{t \rightarrow \infty} x(t) = 0$ . Explain how your answer to part (a) guarantees this.

If  $b$  is positive, then the exponent for  $e$  will be negative ( $-\frac{b}{2}t$ ) in the general solution for  $x(t)$  ( $C_1 e^{at} \cos(bt) + C_2 e^{at} \sin(bt)$ ). This means that both terms will approach zero as  $t$  approaches  $\infty$ ; thus the value of  $x(t)$  will approach zero as well.

Hint: it is necessary to do three different cases.

I am not exactly sure what was meant by this. If you have a different explanation and answer to (a) and (b) that requires three different cases to be explained, then your method is probably what the course head, Robin, had in mind. Otherwise, I believe that the above logic is sound and would suffice.

3. Write a differential equation of the form  $x'' + bx' + cx = 0$  such that if  $x(0) = 1$  and  $x'(0) = 2$  then  $x(t)$  has the property that

(a)  $\lim_{t \rightarrow \infty} x(t) = 0$

$$x'' + 6x' + 9x = 0 \quad x(t) = e^{-3t} + 5te^{-3t}$$

(b)  $\lim_{t \rightarrow \infty} x(t) = \infty$

$$x'' - 7x' + 12 = 0 \quad x(t) = 2e^{3t} - e^{4t}$$

(c)  $\lim_{t \rightarrow \infty} x(t)$  does not exist.

$$x'' - 2x' + 4x = 0 \quad x(t) = e^t \cos(t\sqrt{3}) + \frac{\sqrt{3}}{3} e^t \sin(t\sqrt{3})$$

Note: there are not unique answers to these problems! - The above are just examples!