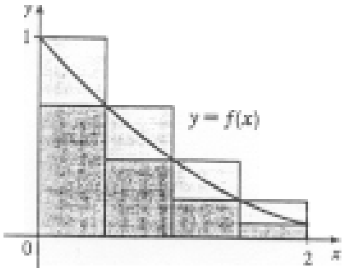


The diagram shows that $L_4 > T_4 > \int_0^2 f(x)\,dx > R_4$, and it appears that M_4 is a bit less than $\int_0^2 f(x)\,dx$. In fact, for any function that is concave upward, it can be shown that

$L_n > T_n > \int_0^2 f(x)\,dx > M_n > R_n$.

(a) Since $0.9540 > 0.8675 > 0.8632 > 0.7811$, it follows that $L_n = 0.9540$, $T_n = 0.8675$, $M_n = 0.8632$, and $R_n = 0.7811$.

(b) Since $M_n < \int_0^2 f(x)\,dx < T_n$, we have $0.8632 < \int_0^2 f(x)\,dx < 0.8675$.

13. $f(y) = \dfrac{1}{1+y^5}$, $\Delta y = \dfrac{3-0}{6} = \dfrac{1}{2}$

(a) $T_6 = \frac{1}{2\cdot 2}\left[f(0) + 2f(\frac{1}{2}) + 2f(\frac{2}{2}) + 2f(\frac{3}{2}) + 2f(\frac{4}{2}) + 2f(\frac{5}{2}) + f(3)\right] \approx 1.064275$

(b) $M_6 = \frac{1}{2}\left[f(\frac{1}{4}) + f(\frac{3}{4}) + f(\frac{5}{4}) + f(\frac{7}{4}) + f(\frac{9}{4}) + f(\frac{11}{4})\right] \approx 1.067416$

(c) $S_6 = \frac{1}{2\cdot 3}\left[f(0) + 4f(\frac{1}{2}) + 2f(\frac{2}{2}) + 4f(\frac{3}{2}) + 2f(\frac{4}{2}) + 4f(\frac{5}{2}) + f(3)\right] \approx 1.074915$

23. $\Delta x = (4-0)/4 = 1$

(a) $T_4 = \frac{1}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)] \approx \frac{1}{2}[0 + 2(3) + 2(5) + 2(3) + 1] = 11.5$

(b) $M_4 = 1\cdot[f(0.5) + f(1.5) + f(2.5) + f(3.5)] \approx 1 + 4.5 + 4.5 + 2 = 12$

(c) $S_4 = \frac{1}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)] \approx \frac{1}{3}[0 + 4(3) + 2(5) + 4(3) + 1] = 11.\overline{6}$

24. We use Simpson's Rule with $n = 10$ and $\Delta x = \frac{1}{2}$:

$$\begin{aligned}
\text{distance} = \int_0^5 v(t)\,dt \approx S_{10} &= \tfrac{1}{2\cdot 3}[f(0) + 4f(0.5) + 2f(1) + \cdots + 4f(4.5) + f(5)] \\
&= \tfrac{1}{6}[0 + 4(4.67) + 2(7.34) + 4(8.86) + 2(9.73) + 4(10.22) \\
&\qquad + 2(10.51) + 4(10.67) + 2(10.76) + 4(10.81) + 10.81] \\
&= \tfrac{1}{6}(268.41) = 44.735 \text{ m}
\end{aligned}$$

34. Let f be a polynomial of degree ≤ 3; say $f(x) = Ax^3 + Bx^2 + Cx + D$. It will suffice to show that Simpson's estimate is exact when there are two subintervals ($n = 2$), because for a larger even number of subintervals the sum of exact estimates is exact. As in the derivation of Simpson's Rule, we can assume that $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. Then Simpson's approximation is

$$\begin{aligned}
\int_{-h}^{h} f(x)\,dx &\approx \tfrac{1}{3}h[f(-h) + 4f(0) + f(h)] \\
&= \tfrac{1}{3}h\left[(-Ah^3 + Bh^2 - Ch + D) + 4D + (Ah^3 + Bh^2 + Ch + D)\right] \\
&= \tfrac{1}{3}h\left[2Bh^2 + 6D\right] = \tfrac{2}{3}Bh^3 + 2Dh
\end{aligned}$$

The exact value of the integral is

$$\begin{aligned}
\int_{-h}^{h}(Ax^3 + Bx^2 + Cx + D)\,dx &= 2\int_0^h (Bx^2 + D)\,dx \quad \text{[by Theorem 5.5.6(a) and (b)]} \\
&= 2\left[\tfrac{1}{3}Bx^3 + Dx\right]_0^h = \tfrac{2}{3}Bh^3 + 2Dh
\end{aligned}$$

Thus, Simpson's Rule is exact.

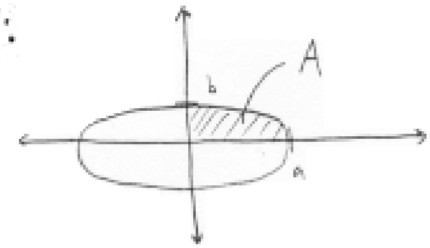
Extra credit:

72. Integrate by parts with $u = (\ln x)^n$, $dv = dx \Rightarrow du = n(\ln x)^{n-1}\cdot\frac{1}{x}\,dx$, $v = x$:
$\int(\ln x)^n\,dx = x(\ln x)^n - \int x\cdot n(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n\int(\ln x)^{n-1}\,dx$. Thus,

$$\begin{aligned}
\int_0^1(\ln x)^n\,dx &= \lim_{t\to 0^+}\int_t^1(\ln x)^n\,dx = \lim_{t\to 0^+}\left[x(\ln x)^n\right]_t^1 - n\lim_{t\to 0^+}\int_t^1(\ln x)^{n-1}\,dx \\
&= -\lim_{t\to 0^+}\frac{(\ln t)^n}{1/t} - n\int_0^1(\ln x)^{n-1}\,dx = -n\int_0^1(\ln x)^{n-1}\,dx,
\end{aligned}$$

by repeated application of l'Hospital's Rule. We want to prove that $\int_0^1(\ln x)^n\,dx = (-1)^n n!$ for every positive integer n. For $n = 1$, we have $\int_0^1(\ln x)^1\,dx = (-1)\int_0^1(\ln x)^0\,dx = -\int_0^1 dx = -1$ (or $\int_0^1 \ln x\,dx = \lim_{t\to 0^+}[x\ln x - x]_t^1 = -1$). Assuming that the formula holds for n, we find that
$\int_0^1(\ln x)^{n+1}\,dx = -(n+1)\int_0^1(\ln x)^n\,dx = -(n+1)(-1)^n n! = (-1)^{n+1}(n+1)!$.
This is the formula for $n+1$. Thus, the formula holds for all positive integers n by induction.

Plus:



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \quad 1 \quad \Rightarrow \quad x\ b^2 \quad a^2y^2 \quad a^2b^2$$

$$\Rightarrow \quad y^2 \quad \frac{a^2b^2 \quad x^2b}{a^2}$$

$$y \quad \pm\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}$$

$$A(\text{ellipse}) \quad 4A_1 \quad 4 \int_{x=0}^{x=a} \underbrace{\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}}_{y} \, dx$$

$$\frac{4b}{a} \quad \sqrt{a^2 - x^2} \quad dx$$

let $x = a\sin\theta$
$dx = a\cos\theta\, d\theta$

$$\frac{4b}{a} \int_{\theta=0}^{\theta=\pi/2} \sqrt{a^2 - a^2\sin^2\theta} \quad a\cos\theta\, d\theta \qquad 4b \quad \sqrt{\ ^2\underbrace{(1-\sin^2\theta)}_{\cos^2\theta}} \quad \cos\theta\, d\theta$$

$$4b \quad a\,|\cos\theta| \quad \cos\theta \quad d\theta \qquad 4ab\int \cos^2\theta \; d\theta$$

(b/c $\cos\theta \geq 0$ in $0 \leq \theta < \frac{\pi}{2}$)

$$4ab \quad \frac{1}{2} + \frac{\cos 2\theta}{2} \quad d\theta \qquad 2ab\left[\theta + \frac{1}{2}\sin 2\theta\right]^{\pi/2}$$

$$= 2ab\left[\left(\frac{\pi}{2} + 0\right) \quad (0+0)\right] \quad \pi ab$$