Problem Set #20

Section 6.1 #1, 4, 16, 22, 23, 24, 25, 27 (extra credit)

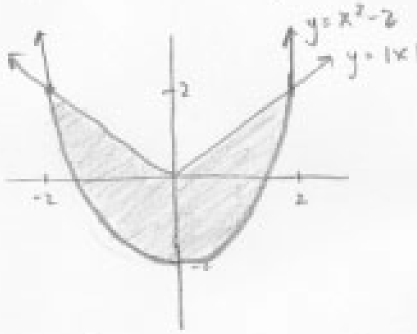
① $A = \int_0^4 (y_{top} - y_{bottom})\,dx = \int_0^4 [(5x - x^2) - x]\,dx = \int_0^4 (4x - x^2)\,dx$

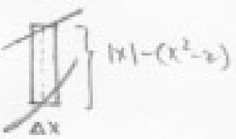
$= \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \left(32 - \frac{64}{3}\right) - (0) = \boxed{\frac{32}{3}}$

④ $A = \int_0^3 (x_{right} - x_{left})\,dy = \int_0^3 [(2y - y^2) - (y^2 - 4y)]\,dy = \int_0^3 (-2y^2 + 6y)\,dy$

$= \left[-\frac{2}{3}y^3 + 3y^2\right]_0^3 = (-18 + 27) - (0) = \boxed{9}$

⑯ $y = |x|$ and $y = x^2 - 2$





using symmetry

$A = \int_{-2}^{2} [|x| - (x^2 - 2)]\,dx = 2\int_0^2 [x - (x^2 - 2)]\,dx = 2\int_0^2 (-x^2 + x + 2)\,dx$

$= 2\left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x\right]_0^2 = 2\left(-\frac{8}{3} + 2 + 4\right) = 2\left(\frac{10}{3}\right) = \boxed{\frac{20}{3}}$

㉒ note that $\int_0^x v_A(t)\,dt = S_A(t)$ for $t = 0$ to $t = x$, where $v_A(t)$ is the velocity of car A and $S_A(t)$ is the displacement of car A.

a) After one minute, area under curve A > area under curve B, so A is ahead after one minute.

b) The area of the shaded region is $S_A(1) - S_B(1)$, the distance by which A is ahead of B after one minute

c) After two minutes, it still appears that A is ahead of B since the area under curve A > area under curve B.

d) Estimating from the graph, it appears that the areas under curves A and B are equal when $t \approx 2.2$ minutes.

(23) Simpson's Rule with $n=8$ and $\Delta x = 2$

$$\text{Area} = \frac{2}{3}\left[0 + 4(6.2) + 2(7.2) + 4(6.8) + 2(5.6) + 4(5.0) + 2(4.8) + 4(4.8) + 0\right]$$

$$= \frac{2}{3}(126.4) = 84.2\bar{6} \approx \boxed{84 \text{ m}^2}$$ (answers may vary)

(24) The area represents the change in revenue minus the change in cost from $x=50$ to $x=100$. In other words, the region is the increase in profit as production is increased from 50 to 100 units

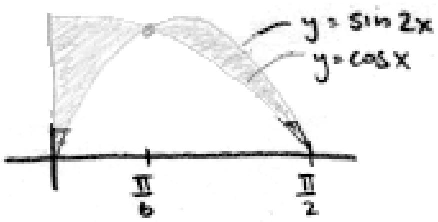
Midpoint Rule with $n=5$ and $\Delta x = 10$

$$M_5 = \Delta x\left[(R'(55)-C'(55)) + (R'(65)-C'(65)) + (R'(75)-C'(75)) + (R'(85)-C'(85)) + (R'(95)-C'(95))\right]$$

$$\approx 10\left[(2.4-.85)+(2.2-.90)+(2.0-1.0)+(1.8-1.1)+(1.6-1.25)\right]$$

$$= 10(4.9) = 49 \Rightarrow \boxed{\$49{,}000}$$ (answers may vary)

(25) $y = \cos x$ and $y = \sin 2x$



solve for intersection:

$\cos x = \sin 2x = 2\sin x \cos x$

$\cos x(2\sin x - 1) = 0$

$2\sin x =$ or $\cos x = 0$

$x = \frac{\pi}{6}$ or $\frac{\pi}{2}$

$$A = \int_0^{\frac{\pi}{6}} (\cos x - \sin 2x)\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin 2x - \cos x)\,dx$$

$$\left[\sin x + \tfrac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}} + \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$\left(\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2}\right) - \left(0 - \tfrac{1}{2}\right) + \left(-\tfrac{1}{2}\cdot(-1) - \right) - \left(-\tfrac{1}{2}\cdot\tfrac{1}{2} - \tfrac{1}{2}\right)$$

$$\frac{3}{4} - \frac{1}{2} - \frac{1}{2} + \frac{3}{4} = \boxed{\frac{1}{2}}$$

(27) extra credit

large circle: $x^2+y^2=R^2$

small circle $x^2+(y-b)^2=r^2$ $\Bigg\}$ $b=\sqrt{R^2-r^2}$

$$A=\int^r \left[\left(b+\sqrt{r^2-x^2}\right)-\sqrt{R^2-x^2} \right] dx$$

$$\left(b+\sqrt{r^2-x^2}-\sqrt{R^2-x^2}\right) dx$$

$$2\int_0^r b\,dx + 2\int_0^r \sqrt{r^2-x^2}\,dx + 2\int_0^r \sqrt{R^2-x^2}\,dx$$

1) $2\int_0^r b\,dx = 2br = 2r\sqrt{R^2-r^2}$

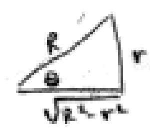
2) $2\int_0^r \sqrt{r^2-x^2}\,dx = 2$(area of quarter circle of radius r) $= 2\left(\frac{1}{4}\pi r^2\right)$

3) $2\int_0^r \sqrt{R^2-x^2}\,dx$ $\quad x=R\sin\theta \quad dx=R\cos\theta\,d\theta$

$$=2\int_0^r \sqrt{R^2-R^2\sin^2\theta}\cdot R\cos\theta\,d\theta = 2\int_0^{\sin^{-1}(\frac{r}{R})} R^2\cos^2\theta\,d\theta = 2R^2\int_0^{\sin^{-1}(\frac{r}{R})} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$=R^2\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\sin^{-1}(\frac{r}{R})} = R^2\sin^{-1}\left(\frac{r}{R}\right)+\frac{R^2}{2}\sin 2\left(\sin^{-1}\left(\frac{r}{R}\right)\right)$$

$$R^2\sin^{-1}\left(\frac{r}{R}\right)+R^2\sin\left(\sin^{-1}\left(\frac{r}{R}\right)\right)\cos\left(\sin^{-1}\left(\frac{r}{R}\right)\right)$$



$$R^2\sin^{-1}\left(\frac{r}{R}\right)+R^2\left(\frac{r}{R}\right)\left(\frac{\sqrt{R^2-r^2}}{R}\right) = R^2\sin^{-1}\left(\frac{r}{R}\right)+r\sqrt{R^2-r^2}$$

$$A=\left(2r\sqrt{R^2-r^2}\right)+\left(\frac{1}{2}\pi r^2\right)-\left(R^2\sin^{-1}\left(\frac{r}{R}\right)+r\sqrt{R^2-r^2}\right)$$

$$\boxed{r\sqrt{R^2-r^2}+\frac{1}{2}\pi r^2-R^2\sin^{-1}\left(\frac{r}{R}\right)}$$