Mathematics 1b - Solution Set for PS 17

Section 7.1 # 1, 3, 4, 6, 9, 11, 12, 14

1.$y = x - x^{-1} \Rightarrow y' = 1 + x^{-2}$. To show that y is a solution of the differential equation, we will substitute the expressions for y and y' in the left-hand side of the equation and show that the left-hand side is equal to the right-hand side.

$$LHS = xy' + y = x(1 + x^{-2}) + (x - x^{-1}) = x + x^{-1} + x - x^{-1} = 2x = RHS$$

3. (a) $y = \sin kt \Rightarrow y' = k \cos kt \Rightarrow y'' = -k^2 \sin kt$
$y'' + 9y = 0 \Rightarrow -k^2 \sin kt + 9 \sin kt = 0 \Rightarrow (9 - k^2) \sin kt = 0$[for all t] $\Rightarrow 9 - k^2 = 0 \Rightarrow k = \pm 3$
(b) $y = A \sin kt + B \cos kt \Rightarrow y' = Ak \cos kt - Bk \sin kt \Rightarrow y'' = -Ak^2 \sin kt - Bk^2 \cos kt$
$y'' + 9y = 0 \Rightarrow -Ak^2 \sin kt - Bk^2 \cos kt + 9(A \sin kt + B \cos kt) = 0 \Rightarrow (9 - k^2)[A \sin kt + B \cos kt] = 0$
The final equation is true for all values of A and B if $k = \pm 3$

4. $y = e^{rt} \Rightarrow y' = re^{rt} \Rightarrow y'' = r^2 e^{rt}$.
$y'' + y' - 6y = 0 \Rightarrow r^2 e^{rt} + re^{rt} - 6e^{rt} = 0 \Rightarrow e^{rt}(r^2 + r - 6) = 0$.
e^{rt} is never zero, so $(r^2 + r - 6) = 0 \Rightarrow (r + 3)(r - 2) = 0 \Rightarrow r = -3\text{or}2$

6. (a)$y = Ce^{x^2/2} \Rightarrow y' = Ce^{x^2/2}(2x/2) = xCe^{x^2/2} = xy$
(b) See graph p. 520 of solution manual.
(c) $y(0) = 5 \Rightarrow Ce^0 = 5 \Rightarrow C = 5$, so the solution is $y = 5e^{x^2/2}$
(d) $y(1) = 2 \Rightarrow Ce^{1/2} = 2 \Rightarrow C = 2e^{-1/2}$, so the solution is $y = 2e^{-1/2}e^{x^2/2} = 2e^{(x^2-1)/2}$

9. (a) $\frac{dP}{dt} = 1.2P(1 - P/4200)$.Now $\frac{dP}{dt} > 0 \Rightarrow 1 - P/4200 > 0$ [assuming that $P > 0$] $\Rightarrow P/4200 < 1 \Rightarrow P < 4200 \Rightarrow$the population is increa sing for $0 < P < 4200$.
(b) $\frac{dP}{dt} < 0 \Rightarrow P > 4200$
(c) $\frac{dP}{dt} = 0 \Rightarrow P = 4200\text{or } P = 0$

11. First graph: This function is increasing *and* also decreasing. But $dy/dt = e^t(y - 1)^2 \geq 0$ for all t, implying that the graph of the solution of the differential equation cannot be decreasing on any interval.
Second graph: When $y = 1, dy/dt = 0$, but the graph does not have a horizontal tangent line.

12. The correct equation is C.
A is not correct, because $y' = 1 + xy > 1$for points in the first quadrant, but from the graph we can see tha t $y' < 0$ for some points in the first quadrant.
B is not correct, because $y' = -2xy = 0$ when $x = 0$, but we can see that $y' > 0$ for $x = 0$.
So C is correct by elimination, and because it equals 1 when $x = 0$, and is greater than 1 in the second quadrant, and is less that one in the first quadrant.

14. (a) The coffee cools most quickly as soon as it is removed from the heat source. The rate of cooling decrea ses toward 0 since the coffee approaches room temperature.
(b) $\frac{dy}{dt} = k(y - R)$,where k is a proportionality constant, y is the temperature of the coffee, and R is the room temperature. The inital condition is $y(0) = 95^0C$. The answer and the model support each other because as y approaches $R, dy/dt$ approaches 0, so the model seems appropriate.
(c) See the graph on page 521 in the solutions manual. It is a decreasing graph with y-intercept at 95 and a ho rizontal asymptote at 20.

Section 7.2 #1

1.(a) See graph on page 522 of the solutions manual.
(b) It appears that the constant functions $y = 0, y = -2$,and $y = 2$ are equilibrium solutions. Note that these three values of y satisfy the given differential equation $y' = y(1 - (1/4)y^2)$.

Handout p.990 # 1, 2, 4

1. (a) $\frac{dM}{dt} = 0.04M$
(b) $\frac{dM}{dt} = 1000 + 0.04M$

2. $\frac{dP}{dt} = kP(N - P)$,where k is the positive constant of proportionality.

4. $\frac{dG}{dt} = T - kG$,where k is the positive constant of proportionality.

Problems 1-3 from Handout F

1. (a) Take the sum of the number of calories in succesively larger spherical "shells." The equation for the surface area of a spherical shell is $4\pi r^2$,so:

$$\# \text{ of Calories} = 4\pi \int_0^R x^2 \rho(x) dx$$

(b) Take the sum of the number of calories in circular slices. Note that in this problem x is the distance from the top of the mold, which would normally be on our $y-$axis. So, if we take the height of each slice to be dx, then the radii are given by $y = \sqrt{R^2 - x^2}$, so the number of calories in each slice is given by $\pi(R^2 - x^2)\delta(x)dx$, and the integral is:

$$\# \text{ of Calories} = \pi \int_0^R (R^2 - x^2)\delta(x) dx$$

2. The graph of $f(x)$ can never be both increasing and concave up because, if $f(x)$ is increasing, then $y' > 0$. Since $y' + y'' = -x^2 \Rightarrow y' = -x^2 - y''$, then $-x^2 - y'' > 0 \Rightarrow y'' < -x^2$if $f(x)$is increasing. Since $-x^2$ is always less than or equal to zero, $y'' < 0$ if $f(x)$ is increasing, so it is concave down.
3. (a) Note that if $f''(x) = 0, f(x)$ is a linear function $Ax + B$.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{b-a}[A\frac{x^2}{2} + Bx]_a^b = \frac{1}{b-a}[\frac{Ab^2}{2} + Bb - \frac{Aa^2}{2} - Ba],$$

$$= \frac{1}{b-a}[\frac{A(b^2 - a^2)}{2} + B(b-a)] = A(\frac{a+b}{2}) + B = f(\frac{a+b}{2}),$$

which is the value of the function at the midpoint of the interval.
(b) If $f''(x) > 0$, then the rate at which $f(x)$ changes is getting faster. Hence, points with greater y valu e will lie farther on the x axis, and the average will be skewed to the right, making it greater than the midpo int.
(c) The same idea as (b), only in the other direction.