# *Mathematics 1b - Solution Set for PS 2*

**Problem Set # 2**

Do: §8.2 # 18, 20, 34, 48, 51

18) $\sum_{n=1}^{\infty} \frac{3}{n} = 3 * \sum_{n=1}^{\infty} \frac{3}{n}$, which diverges because its partial sums are 3 times the corresponding partial sum of the harmonic seris $\sum_{n=1}^{\infty} \frac{3}{n}$, which diverges.

20) $\sum_{n=1}^{\infty} (\frac{(n+1)^2}{n(n+2)})$ diverges because of the Test for Divergence.
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n} = 1$, not 0.

34) $\sum_{n=0}^{\infty} (2^n(x+1)^n) = \sum_{n=0}^{\infty} [2(x+1)]^n$.
$r = 2(x+1)$, so the series converges if $|r| < 1$:
$|2(x+1)| < 1 \longrightarrow |2x+2| < 1 \longrightarrow -1 < 2x+2 < 1 \longrightarrow -3 < 2x < -1 \longrightarrow -\frac{3}{2} < x < -\frac{1}{2}$.
The sum would be $\frac{1}{1-2(x+1)} = \frac{1}{-1-2x} = \frac{-1}{2x+1}$.

48) If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$ by Theorem 6, so $\lim_{n\to\infty} \frac{1}{a_n}$ cannot be zero. Thus, $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is divergent by the Test for Divergence.

51) The partial sums $s_n$ form an increasing sequence, since $s_n - S_{n-1} = a_n > 0$ for all $n$. Also, the sequence $s_n$ is bounded since $s_n \leq 1000$ for all $n$. So by the Monotonic Sequence Theorem, the sequence of partial sums converges and the series is convergent.

*Think, and think again:* Write out the first few terms of the series $\sum_{k=1}^{\infty} (\frac{1}{2^k+k})$. (This series is not a geometric series.) Now write out the first few terms of the geometric series $\sum_{k=1}^{\infty} (\frac{1}{2^k})$. By comparing the terms of the two series, determine whether or not the former series converges. Explain your reasoning in words carefully and clearly. Your answer will form the launching pad for the next class.

$\sum_{k=1}^{\infty} (\frac{1}{2^k+k}) = \frac{1}{3} + \frac{1}{6} + \frac{1}{11} + \frac{1}{20} + ...$
$\sum_{k=1}^{\infty} (\frac{1}{2^k}) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$
In general, $\frac{1}{2^k} > \frac{1}{2^k+k}$ for any $k > 0$. Thus, $\sum_{k=1}^{\infty} (\frac{1}{2^k+k})$ must converge because $\sum_{k=1}^{\infty} (\frac{1}{2^k})$ is a geometric with $r = \frac{1}{2} < 1$ and with partial sums that are always greater than corresponding partial sums of the non-geometric series. Thus, if the geometric converges and both series are constantly increasing (as these both are), the non-geometric series must also converge.

Extra credit # 52.

52) a) $\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}} = \frac{f_n f_{n+1} - f_n f_{n-1}}{f_n^2 f_{n-1} f_{n+1}} = \frac{f_{n+1} - f_{n-1}}{f_n f_{n-1} f_{n+1}} = \frac{(f_{n-1}+f_n) - f_{n-1}}{f_n f_{n-1} f_{n+1}} = \frac{1}{f_{n-1} f_{n+1}}$

b) $\sum_{n=2}^{\infty} (\frac{1}{f_{n-1}f_{n+1}}) = \sum_{n=2}^{\infty} (\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}) = \lim_{n\to\infty}((\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3}) + (\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4}) + (\frac{1}{f_3 f_4} - \frac{1}{f_4 f_5}) + ... + (\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}))$

$= \lim_{n\to\infty} (\frac{1}{f_1 f_2} - \frac{1}{f_n f_{n+1}} = \frac{1}{f_1 f_2} - 0 = \frac{1}{1*1} = 1$

c) $\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty} (\frac{f_n}{f_{n-1}f_n} - \frac{f_n}{f_n f_{n+1}}) = \sum_{n=2}^{\infty} (\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}) = \lim_{n\to\infty}[(\frac{1}{f_1} - \frac{1}{f_3}) + (\frac{1}{f_2} - \frac{1}{f_4}) + (\frac{1}{f_3} - \frac{1}{f_5}) + (\frac{1}{f_4} - \frac{1}{f_6}) + ... + (\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}})]$

$\lim_{n\to\infty} (\frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_n} - \frac{1}{f_{n+1}}) = 1 + 1 - 0 - 0 = 2.$

# *Mathematics 1b - Solution Set for PS 3*

**Problem Set # 3**

Do: §8.3 # 1, 2, 3, 4, 10, 19, 20.

1) The picture shows that $a_2 = \frac{1}{2^{1.3}} < \int_1^2 \frac{1}{x^{1.3}}dx$, $a3 = \frac{1}{3^{1.3}} < \int_2^3 \frac{1}{x^{1.3}}dx$, and so on, so $\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}}dx$. The integral converges because it is a p-series with $p = 1.3 > 1$, so the series converges.

2) From the first figure, we see that $\int_1^6 f(x)dx < \sum_{i=1}^{5} a_i$. From the second figure, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)dx$. Therefore, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)dx$.

3a) We cannot say anything because $\sum a_n$ is not bound above by a series known to converge. Therefore, $\sum a_n$ can either converge or diverge. (Answers can vary - use your judgment.)
3b) If $a_n < b_n$ for all $n$, then $\sum a_n$ is convergent because it is always positive and bound by $\sum b_n$, a series that has greater partial sums and that converges. (part (i) of Comparison Test)

4a) If $a_n > b_n$ for all $n$, then $\sum a_n$ is divergent. (part (ii) of Comparison Text)
4b) We cannot say anything about $\sum a_n$ because it is not bound below by a series known to diverge. Therefore, $\sum a_n$ can either converge or diverge. (Again, use your judgment.)

10) $\frac{1}{2n-1} > \frac{1}{2n} = \frac{1}{2}\frac{1}{n}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$, a series that diverges and has lower corresponding terms. (part (ii) of Comp. Test)

19) $\frac{n+1}{n^2} > \frac{n}{n^2} = \frac{1}{n}$ for all $n \geq 1$. Thus, $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$. (part (ii) of Comp. Test)

20) $\frac{4+3^n}{2^n} > \frac{3^n}{2^n} = (\frac{3}{2})^n$ for all $n \geq 1$. Thus, $\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$ diverges by comparison with $\sum_{n=1}^{\infty} (\frac{3}{2})^n$, a geometric series that diverges ($r = \frac{3}{2} > 1$).

Plus:

*Problem on p−series*

In this problem you will learn about a family of series known as $p-$series. A $p-$series is a series of the form $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$.

1. If $p < 0$ then the series diverges by the $n$th term test. Explain.

   I am actually not sure what the $n$th term test is, but if $p < 0$, the series would be the same as $\sum_{n=1}^{\infty} n^{-p}$ (with -p being positive), which diverges based on the Test for Divergence because $\lim_{n\to\infty} n^{positivevalue} \neq 0$.

2. Show that if $p > 1$ then $\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx$ is finite.

   $\lim_{b\to\infty} \int_1^b \frac{1}{x^p} = \lim_{b\to\infty} \frac{-1}{(p-1)x^{p-1}}|_1^b = \lim_{b\to\infty} \frac{-1}{(p-1)b^{p-1}} - \frac{-1}{(p-1)(1)}$
   If $(p-1) < 0 or p < 1$, then the above integral would approach infinity because the exponent of $b$ would be negative and would bring $b$ to the numerator (with a positive exponent). This would approach infinity as b approaches infinity.

   If $(p-1) = 0 or p = 1$, then the integral of $\frac{1}{x}$ would be $\lim_{b\to\infty} \ln x|_1^b$, which is infinite (because $\lim_{b\to\infty} \ln b = \infty$).

   If $(p-1) > 0 or p > 1$, the definite integral is finite because $b$ remains in the denominator and causing that term to approach zero as b approaches infinity. Thus, if $p > 1$, the definite integral is finite.

   (NOTE: Students only need to show the last third, since that is all that was asked. I gave more in case some students explain all ranges.)

Show that if $0 < p < 1$ then $\lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx = \infty$.

Along the same lines as above, if $0 < p < 1$, then $(p-1) < 0$ and $\lim_{b\to\infty} \frac{1}{b^{p-1}} = \infty$. Thus, $\lim_{b\to\infty} \int_1^b \frac{1}{x^p} = \lim_{b\to\infty} \frac{-1}{(p-1)x^{p-1}}\Big|_1^b = \lim_{b\to\infty} \frac{-1}{(p-1)b^{p-1}} - \frac{-1}{(p-1)(1)} = \infty - \frac{-1}{(p-1)(1)}$. Because the second term is finite, the difference is infinite. Thus, $\lim_{b\to\infty} \int_1^b \frac{1}{x^p} = \infty$ if $0 < p < 1$.

(You'll need to break this into two cases. Think about why.)

I don't think that we need to split it into two cases - am I missing something?)

Conclude that $\int_1^\infty \frac{1}{x^p}\,dx$ diverges for $0 < p < 1$ and converges for $p > 1$.

Based on the Integral Test, $\int_1^\infty \frac{1}{x^p}$ converges when $p > 1$ because $\int_1^\infty \frac{1}{x^p}$ is convergent for those values of $p$. Likewise, $\int_1^\infty \frac{1}{x^p}$ diverges when $0 < p < 1$ because $\int_1^\infty \frac{1}{x^p}$ is divergent for those values of $p$.

3. Conclude from your work in parts (a) and (b) that $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$ converges if $p > 1$ and diverges if $p < 1$.

It was shown above that $\sum_{n=1}^\infty \frac{1}{n^p}$ diverges if $p < 0$ and if $0 < p < 1$. If $p = 0$, the series becomes $\sum_{n=1}^\infty 1$, which diverges as well due to the Divergence Test. Thus, $\sum_{n=1}^\infty \frac{1}{n^p}$ diverges if $p < 1$ and converges if $p > 1$.