Extra credit: Handout G

#30

#1 a) $\frac{dv}{dt} = -32 + kv^2$ $\quad v(0)=0$

b) $\frac{dv}{kv^2-32} = dt \Rightarrow \int \frac{dv}{kv^2-32} = t + C \Rightarrow \int \frac{A}{\sqrt{k}v+4\sqrt2} + \frac{B}{\sqrt{k}v-4\sqrt2}\,dv = t+C$

$A=-B,\ 4\sqrt2(A+B)=1 \to A = \frac{-1}{8\sqrt2}\quad B=\frac{1}{8\sqrt2}$

So $\frac{-1}{8\sqrt{2k}}\ln|\sqrt k v+4\sqrt2| + \frac{1}{8\sqrt{2k}}\ln|\sqrt k v - 4\sqrt2| = t+C \to \frac{1}{8\sqrt{2k}}\ln\left|\frac{\sqrt k v-4\sqrt2}{\sqrt k v+4\sqrt2}\right| = t+C$

Initial condition gives $C=0$. So $\boxed{\frac{1}{8\sqrt{2k}}\ln\left|\frac{\sqrt k v - 4\sqrt2}{\sqrt k v+4\sqrt2}\right| = t}$

c) $\left|\frac{\sqrt k v-4\sqrt2}{\sqrt kv+4\sqrt2}\right| = e^{8t\sqrt{2k}} \Rightarrow \left|1-\frac{8\sqrt2}{\sqrt kv+4\sqrt2}\right| = e^{8t\sqrt{2k}}$ We always know $kv^2<32 \Rightarrow \sqrt k v < 4\sqrt2$ (as object does not go upwards). So $\frac{8\sqrt2}{\sqrt kv+4\sqrt2} - 1 = e^{8t\sqrt{2k}} \to \sqrt k v+4\sqrt2 = \frac{8\sqrt2}{1+e^{8t\sqrt{2k}}} \to \boxed{v = \frac{1}{\sqrt k}\left(\frac{8\sqrt2}{1+e^{8t\sqrt{2k}}}-4\sqrt2\right)}$

d) $d(t_0)\ \underbrace{\left|\frac{1}{\sqrt k}\int_0^{t_0}\left(\frac{8\sqrt2}{1+e^{8t\sqrt{2k}}} - 4\sqrt2\right)\right|}_{\text{always negative}} dt = \frac{4\sqrt2}{\sqrt k} t_0 - \frac{8\sqrt2}{\sqrt k}\int_0^{t_0}\frac{1}{1+e^{8t\sqrt{2k}}}\,dt$

come back to this later

$u = 1+e^{8t\sqrt{2k}}$

$du = 8\sqrt{2k}\,e^{8t\sqrt{2k}}\,dt \quad dt = 8\sqrt{2k}(u-1)\,dt$

So $dt = \frac{du}{8\sqrt{2k}(u-1)} \to \frac{-8\sqrt2}{\sqrt k}\int_2^{1+e^{8t_0\sqrt{2k}}} \frac{du}{8\sqrt{2k}\,u(u-1)} \to \frac{-1}{k}\int_2^{1+e^{8t_0\sqrt{2k}}}\frac{du}{u(u-1)} \to \frac1k \ln\left|\frac{u}{u-1}\right|_2^{1+e^{8t_0\sqrt{2k}}}$

$= \frac1k \ln\left|\left(\frac{1}{e^{8t_0\sqrt{2k}}}+1\right)/2\right|$ So final answer is (I hope)

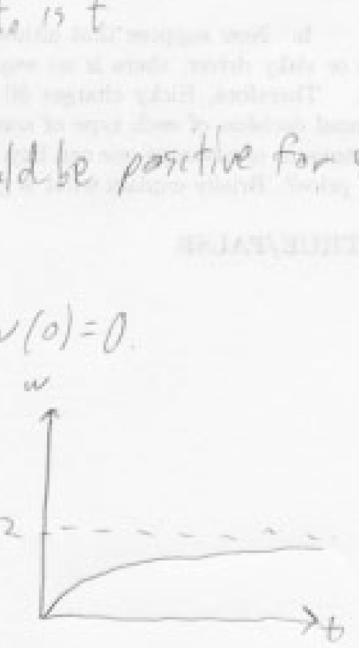
$\boxed{\frac{4\sqrt2}{\sqrt k}t_0 + \frac1k\ln\left|\frac{\frac{1}{e^{8t_0\sqrt{2k}}}+1}{2}\right|}$ (sort of makes sense, as air resistance factor is negative). Here t_0 is t

Non-Extra Credit

#2) Lots of reasons. For instance, y should be positive for $y > \frac{\pi}{2}$, but graph is decreasing.

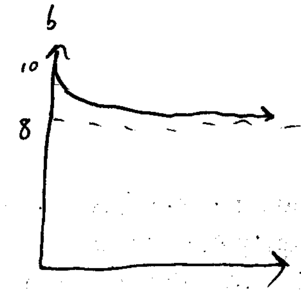
#3) a) $\frac{dw}{dt} = .4 - 2\frac{w}{10} = .4 - .2w.$ $\quad w(0)=0.$

Equilibrium at $w=2$. $w(0)=0$, so



b) $\frac{db}{dt} = 1.6 - \frac{2b}{10} = 1.6 - .2b \quad b(0) = 10$

Equilibrium at $b = 8$. $b(0) = 10$, so



13. (a) If $y = u - 75$, $u(0) = 185 \Rightarrow y(0) = 185 - 75 = 110$, and the initial-value problem is $dy/dt = ky$ with $y(0) = 110$. So the solution is $y(t) = 110e^{kt}$.

(b) $y(30) = 110e^{30k} = 150 - 75 \Rightarrow e^{30k} = \frac{75}{110} = \frac{15}{22} \Rightarrow k = \frac{1}{30} \ln \frac{15}{22}$, so $y(t) = 110e^{\frac{1}{30} t \ln\left(\frac{15}{22}\right)}$ and $y(45) = 110e^{\frac{45}{30} \ln\left(\frac{15}{22}\right)} \approx 62\,°\text{F}$. Thus, $u(45) \approx 62 + 75 = 137\,°\text{F}$.

(c) $u(t) = 100 \Rightarrow y(t) = 25$. $y(t) = 110e^{\frac{1}{30} t \ln\left(\frac{15}{22}\right)} = 25 \Rightarrow e^{\frac{1}{30} t \ln\left(\frac{15}{22}\right)} = \frac{25}{110} \Rightarrow$ $\frac{1}{30} t \ln \frac{15}{22} = \ln \frac{25}{110} \Rightarrow t = \dfrac{30 \ln \frac{25}{110}}{\ln \frac{15}{22}} \approx 116$ min.

8. (a) $P(0) = P_0 = 400$, $P(1) = 1200$ and $K = 10{,}000$. From the solution to the logistic differential equation $P(t) = \dfrac{P_0 K}{P_0 + (K - P_0)e^{-kt}}$, we get $P = \dfrac{400\,(10{,}000)}{400 + (9600)e^{-kt}} = \dfrac{10{,}000}{1 + 24e^{-kt}}$. $P(1) = 1200 \Rightarrow$ $1 + 24e^{-k} = \frac{100}{12} \Rightarrow e^k = \frac{288}{88} \Rightarrow k = \ln \frac{36}{11}$. So $P = \dfrac{10{,}000}{1 + 24e^{-t\ln(36/11)}} = \dfrac{10{,}000}{1 + 24 \cdot (11/36)^t}$.

(b) $5000 = \dfrac{10{,}000}{1 + 24(11/36)^t} \Rightarrow 24\left(\frac{11}{36}\right)^t = 1 \Rightarrow t \ln \frac{11}{36} = \ln \frac{1}{24} \Rightarrow t \approx 2.68$ years.

9. (a) $\dfrac{dP}{dt} = k(P)\left(1 - \dfrac{P}{K}\right) \Rightarrow$

$$\frac{d^2P}{dt^2} = k\left[P\left(-\frac{1}{K}\frac{dP}{dt}\right) + \left(1 - \frac{P}{K}\right)\frac{dP}{dt}\right] = k\frac{dP}{dt}\left(-\frac{P}{K} + 1 - \frac{P}{K}\right)$$
$$= k\left[kP\left(1 - \frac{P}{K}\right)\right]\left(1 - \frac{2P}{K}\right) = k^2P\left(1 - \frac{P}{K}\right)\left(1 - \frac{2P}{K}\right)$$

(b) P grows fastest when P' has a maximum, that is, when $P'' = 0$. From part (a), $P'' = 0 \Leftrightarrow P = 0, P = K$, or $P = K/2$. Since $0 < P < K$, we see that $P'' = 0 \Leftrightarrow P = K/2$.

▲ TRUE–FALSE QUIZ ▲

Since $y^4 \geq 0$, $y' = -1 - y^4 < 0$ and the solutions are decreasing functions. True

$y = \dfrac{\ln x}{x} \Rightarrow y' = \dfrac{1 - \ln x}{x^2}$. True

LHS $= x^2y' + xy = x^2 \cdot \dfrac{1 - \ln x}{x^2} + x \cdot \dfrac{\ln x}{x} = (1 - \ln x) + \ln x = 1 =$ RHS, so $y = \dfrac{\ln x}{x}$ is a solution of $x^2y' + xy = 1$.

$x + y$ cannot be written in the form $g(x)f(y)$. False

$y' = 3y - 2x + 6xy - 1 = 6xy - 2x + 3y - 1 = 2x(3y - 1) + 1(3y - 1) = (2x + 1)(3y - 1)$, so y' can be written in the form $g(x)f(y)$. True

By comparing $\dfrac{dy}{dt} = 2y\left(1 - \dfrac{y}{5}\right)$ with the logistic differential equation (7.5.1), we see that the carrying capacity is 5; that is, $\lim\limits_{t\to\infty} y = 5$. True