

**Solution to TTh PS 24**

1. Solve the following differential equations for  $y(t)$ . Give the general solution.

a)  $y'' + 6y' = 7y$

$$y(t) = C_1 e^{-7t} + C_2 e^t$$

b)  $y'' + 6y' + 9y = 0$

$$y(t) = C_1 e^{-3t} + C_2 t * e^{-3t}$$

c)  $y'' + 5y' + 6y = 0$

$$y(t) = C_1 e^{-2t} + C_2 e^{-3t}$$

2. For each of the differential equations in the problem above, suppose that the initial conditions are  $y(0) = -2$  and  $y'(0) = 0$ .

(i) Use the initial conditions to find  $y(t)$ .

a)  $y(t) = C_1 e^{-7t} + C_2 e^t$

$$C_1 + C_2 = -2$$

$$-7C_1 + C_2 = 0$$

$$C_1 = -\frac{1}{4} \quad C_2 = -\frac{7}{4}$$

$$y(t) = -\frac{1}{4} e^{-7t} + -\frac{7}{4} e^t$$

b)  $y(t) = C_1 e^{-3t} + C_2 t * e^{-3t}$

$$C_1 = -2$$

$$-3C_1 + C_2 = 0$$

$$C_2 = -6$$

$$y(t) = -2e^{-3t} - 6t * e^{-3t} = -e^{-3t}(2 + 6t)$$

c)  $y(t) = C_1 e^{-2t} + C_2 e^{-3t}$

$$C_1 + C_2 = -2$$

$$-2C_1 + -3C_2 = 0$$

$$C_1 = -6 \quad C_2 = 4$$

$$y(t) = -6e^{-2t} + 4e^{-3t}$$

(ii) Find  $\lim_{t \rightarrow \infty} y(t)$ .

a)  $-\infty$

b) 0

c) 0

3. Interpret  $x(t)$  as the position of a mass on a spring at time  $t$  where  $x(t)$  satisfies

$$x'' + 4x' + 3x = 0.$$

Suppose the mass is pulled out, stretching the spring one unit from its equilibrium position, and given an initial velocity of +2 units per second.

(a) Find the position of the mass at time  $t$ .

$$x(t) = -\frac{3}{2} e^{-3t} + \frac{5}{2} e^{-t}$$

(b) Determine whether or not the mass ever crosses the equilibrium position of  $x = 0$ .

$$0 = -\frac{3}{2} e^{-3t} + \frac{5}{2} e^{-t}$$

$$5e^{-t} = 3e^{-3t}$$

$$\frac{e^{-t}}{e^{-3t}} = \frac{3}{5}$$

$$e^{2t} = \frac{3}{5}$$

$$t = \ln(\frac{3}{5})/2 \text{ YES}$$

(c) When (at what time) is the mass furthest from its equilibrium position? Approximately how far from the equilibrium position does it get?

$$x'(t) = \frac{9}{2} e^{-3t} - \frac{5}{2} e^{-t}$$

$$0 = \frac{9}{2} e^{-3t} - \frac{5}{2} e^{-t}$$

$$9e^{-3t} = 5e^{-t}$$

$$\frac{9}{5} = e^{2t}$$

$$t = \ln(\frac{9}{5})/2$$

$$x(\ln(\frac{9}{5})/2) = 1.242259987...$$

4. (a) Suppose that  $x(t) = C_1e^{at} + C_2e^{bt}$ . Show that  $x(t) = 0$  at most once. Find the value of  $t$  for which  $x(t) = 0$  if such a value exists.

$$0 = C_1e^{at} + C_2e^{bt}$$

$$-\frac{C_2}{C_1} = e^{(a-b)t}$$

$$\ln\left(-\frac{C_2}{C_1}\right) = (a-b)t$$

$$t = \ln\left(-\frac{C_2}{C_1}\right)/(a-b)$$

- (b) Suppose that  $x(t) = C_1e^{at} + C_2te^{at}$ . Show that  $x(t) = 0$  at most once. Find the value of  $t$  for which  $x(t) = 0$  if such a value exists.

$$0 = C_1e^{at} + C_2te^{at}$$

$$C_1 = -C_2t$$

$$t = -\frac{C_1}{C_2}$$

- (c) Conclude from parts (a) and (b) that if the characteristic equation of  $x'' + bx' + cx = 0$  has either one real root or two real roots then the differential equation cannot model a mass at the end of a spring in the scenario that the mass oscillates back and forth around the equilibrium point.

If the equation has either one or two real roots, then the equation for  $x(t)$  would resemble the equation in part (b) or (c), respectively. Neither equations can describe oscillating motion because such motion would repeatedly pass through the equilibrium point ( $x(t) = 0$ ), not just once (or not at all).

5. Let's try to make sense of the expression  $e^{(a+bi)t}$ , that is,  $e$  raised to a complex number  $a + bi$  where  $i = \sqrt{-1}$ . To do this, first observe that  $e^{(a+bi)t} = e^{at} \cdot e^{bit}$ , where  $a$  and  $b$  are real numbers. The part we must make sense of is  $e^{bit}$ . Use the Maclaurin Series for  $e^x$  to expand  $e^{bit}$ . Gather all terms with  $i$  and all terms without  $i$ . (Factor out the  $i$  from the terms with  $i$ .) Now rewrite  $e^{bit}$  in terms of familiar functions.

$$e^{bit} = 1 + bit + \frac{1}{2}(bit)^2 + \frac{1}{6}(bit)^3 + \frac{1}{24}(bit)^4 + \dots$$

$$e^{bit} = 1 + bit - \frac{1}{2}(bt)^2 - \frac{i}{6}(bt)^3 + \frac{1}{24}(bt)^4 + \dots$$

$$e^{bit} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} (bt)^{2n} + \sum_{n=0}^{\infty} (-1)^n \frac{i}{(2n+1)!} (bt)^{2n+1}$$

$$e^{bit} = \cos(bt) + i \sin(bt)$$

Given your work above, what's  $e^{\pi i}$ ?

$$e^{\pi i} = \cos(\pi) + i \sin(\pi) = -1 + i(0) = -1$$