PROBLEM SET #7

Handout B

$f(x) = x^3 - 4x^2 + 4x \quad \Rightarrow f(0) = 0$

$f'(x) = 3x^2 - 8x + 4 \quad \Rightarrow f'(0) = 4$

$f''(x) = 6x - 8 \quad \Rightarrow f''(0) = -8$

$f^{(3)}(x) = 6 \quad \Rightarrow f^{(3)}(0) = 6$

a) To get the best linear approximation, match the first derivative and y intercept at $x=0$

$$P_1(x) = 4x + 0 = 4x$$

To get the best quadratic approximation, match the first and the second derivative and the y intercept at $x=0$

$$P_2(x) = -4x^2 + 4x + 0 = (-4x^2) + 4x$$

To get the best cubic approximation, match the first, the second, and the third derivative at $x=0$ and the y intercept:

$$P_3 = x^3 - 4x^2 + 4x$$

$\Rightarrow P_3(x) = f(x)$: this should not be a surprise; the function is a cubic, and the best cubic approximation to a cubic function is the function

b) Solve $f(x) = 0 \Rightarrow f(x) = x(x-2)^2 \Rightarrow x_1 = 0 \quad x = 2$

$f'(2) = 3(2^2) - 8(2) + 4 = 0 \Rightarrow x = 2$ is a local min (check with second derivative).

Since a_1 is the value of the first derivative and the value is 0, a_1 must be 0.

(Note: a_2, though, is not simply the value of the second derivative at $x=0$).

$a_4, a_5, \ldots a_n$ must be zero in our approximation of $f(x)$, for $f(x)$ is a cubic and can't be approximated by a higher degree polynomial: <u>4th and higher derivatives are all zeros, thus $a_4, \ldots, a_n$ must be zeros.</u>

c) No matter what x we choose to approximate $f(x)$ around, 4th and higher derivatives of $f(x)$ will be zeros, thus $a_4, \ldots, a_n$ will be zeros.

<u>Series Handout</u>

1. $f(x) = e^{-x}$ $\quad f'(x) = -e^{-x}$ $\quad f''(x) = e^{-x}$ $\quad f^{(3)}(x) = -e^{-x}$ $\quad f^{(4)}(x) = e^{-x}$

$f(0) = 1$ $\quad \underbrace{f'(0) = -1 \quad f''(0) = 1 \quad f^{(3)}(0) = -1}_{\text{Match derivatives}}$ $\quad f^{(4)}(0) = 1$

$\Rightarrow$

$P_1 = 1 - x$	$P_1(0.1) = 0.9$	$P_1(0.3) = 0.7$
$P_2 = 1 - x + \frac{x^2}{2!}$	$P_2(0.1) = 0.905$	$P_2(0.3) = 0.745$
$P_3 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$	$P_3(0.1) = 0.904833$	$P_3(0.3) = 0.7405$
$P_4 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$	$P_4(0.1) = 0.9048375$	$P_4(0.3) = 0.7408375$
	$e^{-0.1} = 0.904837418$	$e^{-0.3} = 0.7408182207$

$\Rightarrow$ higher degree polynomials give better approximations.

2. $f(x) = \ln(1+x)$ $\quad f'(x) = (1+x)^{-1}$ $\quad f''(x) = -(1+x)^{-2}$ $\quad f^{(3)}(x) = +2(1+x)^{-3}$ $\quad f^{(4)}(x) = -6(1+x)^{-4}$

$f(0) = 0$ $\quad \underbrace{f'(0) = 1 \quad f''(0) = -1 \quad f^{(3)}(0) = 2}_{\text{Match derivatives}}$ $\quad f^{(4)}(0) = -6$

2 ctd.

$\Rightarrow P_1 = 0 + x$ $\qquad P_1(0.1) = 0.1$ $\qquad P_1(0.3) = 0.3$

$P_2 = 0 + x - \frac{x^2}{2}$ $\qquad P_2(0.1) = 0.095$ $\qquad P_2(0.3) = 0.255$

$P_3 = 0 + x - \frac{x^2}{2} + \frac{x^3}{3}$ $\qquad P_3(0.1) = 0.095333$ $\qquad P_3(0.3) = 0.264$

$P_4 = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ $\qquad P_4(0.1) = 0.0953083$ $\qquad P_4(0.3) = 0.261975$

$\ln(1.1) = 0.0953102$ $\qquad \ln(1.3) = 0.2623643$

$\Rightarrow$ higher-degree polynomials give better approximations, but within a small range around $x=0$: the farther away from $x=0$, the worse the approximation.

4.

$f(x) = (1+x)^4$ $\qquad f'(x) = 4(1+x)^3$ $\qquad f''(x) = 12(1+x)^2$ $\qquad f^{(3)}(x) = 24(1+x)$ $\qquad f^{(4)}(x) = 24$

$f(0) = 1$ $\qquad f'(0) = 4$ $\qquad f''(0) = 12$ $\qquad f^{(3)}(0) = 24$ $\qquad f^{(4)}(0) = 24$

Match derivatives

$\Rightarrow P_1 = 1 + 4x$ $\qquad P_1(0.1) = 1.4$ $\qquad P_1(0.3) = 2.2$

$P_2 = 1 + 4x + 6x^2$ $\qquad P_2(0.1) = 1.46$ $\qquad P_2(0.3) = 2.74$

$P_3 = 1 + 4x + 6x^2 + 4x^3$ $\qquad P_3(0.1) = 1.464$ $\qquad P_3(0.3) = 2.848$

$P_4 = 1 + 4x + 6x^2 + 4x^3 + x^4$ $\qquad P_4(0.1) = 1.4641$ $\qquad P_4(0.3) = 2.8561$

$(1.1)^4 = 1.4641$ $\qquad (1.3)^4 = 2.8561$

$\Rightarrow P_4$ is equal to $f(x)$, thus it is the best approximation; the higher the degree of P_n, the better the approximation; the farther away from $x=0$, the worse the approximation.

9. At $x=0$:

a_0: $-$ve : at $x=0$, curve of $f(x)$ lies below $y=0$

a_1: $+$ve : curve sloping upward: 1st derivative positive

a_2: $+$ve : curve concave up: 2nd derivative positive

12. a) a_0 : $-$ve : curve lies below $y=0$ at $x=0$

a_1 : 0 : local minimum: first derivative is zero

a_2 : $+$ve : concave up (at local minimum)

b) a_0 : $-$ve : curve lies below $y=0$ at $x=1$

a_1 : $+$ve : curve slopes upward : 1^{st} derivative positive

a_2 : $+$ve : curve concave up

c) a_0 : 0 : curve crosses $y=0$ at $x=2$

a_1 : $+$ve : curve sloping upwards : 1^{st} derivative positive

a_2 : 0 : $x=2$ is the point of inflection

d) a_0 : $+$ve : curve lies above $y=0$

a_1 : $+$ve : curve sloping upwards : 1^{st} derivative positive

a_2 : $-$ve : concave down

13. $f'(x) = (1+x)^{-1}$ $\quad f''(x) = -(1+x)^{-2}$ $\quad f^{(3)}(x) = 2(1+x)^{-3}$ ...... $f^{(n)} = (n-1)!(1+x)^{-n}(-1)^{n}$,

for $n \geq 1$

$$\Rightarrow \ln(1+x)\Big|_{x=0} \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{n+1}x^n}{n} = \sum_{n=1}^{N} \frac{(-1)^{n-1}x^n}{n}$$