Answers (and Partial Solutions) to Review Problems for the First Exam

1. Todd is thinking that a function (the accumulated area function, or the function $A(b) = \int_0^b f(x)\,dx$) that is increasing must increase without bound. This is incorrect. Consider a function like $1 - \frac{1}{b}$. As b increases $1 - \frac{1}{b}$ increases, but it never gets above 1.

   Amani believes that $\int_1^\infty 1/x\,dx$ converges. Convince him that it doesn't by computing $\lim_{b\to\infty} \int_1^b 1/x\,dx$. The rate at which $\ln x$ is increasing ($\frac{1}{x}$) tends towards zero but $\ln x$ nevertheless increases without bound.

2. The integral converges precisely when $p > 1$: indeed, if $p = 1$, $\int_1^\infty \frac{dx}{x} = \ln x|_1^\infty = \ln\infty - \ln 1 = \infty$, divergent. If $p \neq 1$, we have $\int_0^\infty \frac{dx}{x} = \frac{x^{1-p}}{1-p}|_1^\infty = \frac{\infty^{1-p}}{1-p} - \frac{1}{1-p}$. For this to converge we need $1 - p < 0$, i.e. $p > 1$.

3. It converges precisely when $p < 1$: indeed, if $p = 1$, $\int_0^1 \frac{dx}{x} = \ln x|_0^1 = \ln 1 - \ln 0 = 0 - (-\infty) = \infty$, divergent. If $p \neq 1$, $\int_0^1 \frac{dx}{x^p} = \frac{x^{1-p}}{1-p}|_0^1 = \frac{1}{1-p} - \frac{0^{1-p}}{1-p}$; for 0^{1-p} to be finite, we need $1 - p > 0$, i.e. $p < 1$.

4. In view of problems 2 and 3, never.

5. (a) No, the area is not finite.
   (b) Draw a picture. Compute $\int_{-\infty}^{-1} \pi\frac{1}{x^2}\,dx$. You'll get π. Note: you could convert the problem to the problem of computing $\int_1^\infty \pi\frac{1}{x^2}\,dx$ if you prefer.
   (c) Not finite. (Look at $\int_{-\infty}^{-1} 2\pi x\frac{1}{x}\,dx$.)

6. First we must recognize that it is improper. It blows up at $\sqrt{3}$. Indeed, the way to try to do the integral is by partial fractions, so $\frac{1}{x^2-3} = \frac{1}{(x-\sqrt{3})(x+\sqrt{3})} = \frac{A}{x-\sqrt{3}} + \frac{B}{x+\sqrt{3}}$, and solving for A, B we get $\frac{1}{x^2-3} = \frac{\frac{1}{2\sqrt{3}}}{x-\sqrt{3}} - \frac{\frac{1}{2\sqrt{3}}}{x+\sqrt{3}}$. So the function becomes infinite at $\sqrt{3}$, which is in the interval of integration. Proceeding as in the third problem, we get that the interval is divergent – indeed, the integral of any rational function $\int_a^b \frac{P(x)}{(x-a)^n Q(x)}$ (with $P(a) \neq 0$) will be divergent.

7. Notice that it looks hopeless to find an antiderivative for e^{-x^3}, so we cannot evaluate the integral exactly. Nevertheless, the integrand goes to zero very rapidly, so we intuitively expect the integral to converge. To make this idea work, we make a comparison: when $x \geq 1$, then $x^3 > x$, so $e^{x^3} > e^x$ and $e^{-x^3} < e^{-x}$. Thus $\int_1^\infty e^{-x^3}dx \leq \int_1^\infty e^{-x}dx = -e^{-x}|_1^\infty = -(e^{-\infty} - 1) = 1$. So the integral converges (and is less than 1).

8. Since $|\frac{\sin x}{x^2}| \leq \frac{1}{x^2}$, and from the second problem $\int_1^\infty \frac{dx}{x^2} < \infty$, the given improper integral is *convergent*.

9. $\int_0^{50} 2\pi r f(r)\, dr$

10. The amount of garlic for the whole pizza is : $\int_0^7 g(x) 2\pi x\, dx = \ldots = \frac{2}{3}\pi \int_0^7 \frac{3x^2}{(x^3+2)^2}\, dx = \ldots = \frac{2}{3}\pi \int_2^3 45u^{-2}\, du = \ldots = \frac{343}{1035} \approx 1.0411$. Hence for one slice it will be this quantity divided by 6. (We can do this because of the way we slice pizza.

11. (a) using cylindrical shells we get $\int_0^5 \rho(r)(10)(2\pi r)\, dr = \ldots = 26 \int_0^5 \frac{r}{r+1}\, dr$ Now let $u = r + 1$. In the end you'll get $26(5 - \ln 6) \approx 83.4$, so about 83 insects.

    (b) Using half sphere shells you'll get $\int_0^5 \rho(r) \frac{1}{2}(4\pi r^2)\, dr = \ldots = 2.6 \int_0^5 \frac{r62}{r+1}\, dr$. Again, let $u = r + 1$. You'll get, after computation, $2.6 \int_1^6 (u - 2 + \frac{1}{r})\, du = \ldots 2.6(.5u^2 - 2u + \ln u)|_1^6 \approx 24.2$ so there are about 24 insects.

12. (a) $\int_0^{\pi/4} 2\pi x \tan x\, dx$

    (b) $\int_0^{\pi/4} 2\pi(\pi - x) \tan x\, dx$

    (c) $\pi \int_0^{\pi/4} (\tan x + 3)^2 - 9)\, dx$

13. Let $A(x)$ denote the area of the semicircle at x. Then the volume is $V = \int_0^9 A(x)dx$. To find the area, note that the radius of the semicircle is $\sqrt{x}$, so that the area is $\frac{1}{2}\pi R^2 = \frac{1}{2}\pi x$. Thus the volume is $\frac{\pi}{2} \int_0^9 x dx = \frac{81\pi}{4}$.

14. There are several different ways to get $\frac{4}{3}\pi R^3$.

15. Chop this so that you get disks. (You can think of it as a volume of revolution.)

    $V = \pi \int_{r-h}^{r} (r^2 - y^2)\, dy = \pi[r^2 y - \frac{y^3}{3}]|_{r-h}^{r} = \ldots = \frac{1}{3}\pi h^2(3r - h)$

16. The volume is $\frac{\pi}{3}(r^2 h + R^2 h + Rrh)$.

17. The first thing to do is to draw a picture showing the two points of intersection of the parabola and the line. To find those points analytically, we set $y^2 = x = 2y + 3$ and solve to get $y = 3$ or $y = -1$. Thus the two intersection points are $(1, -1)$ and $(9, 3)$. Notice that if we integrate with respect to x (and use washers) we would have to break up the integral into two pieces: from $x = 0$ to 1 the parabola is the bottom function, while from $x = 1$ to 9 the line is the bottom function. But notice that it is always the case that the line is farther to the right of the parabola, meaning that if we integrate with respect to y (and use shells) we can do the integral all at once. The integral we get is $2\pi \int_{-1}^{3} y[(2y+3) - y^2]dy = \frac{32\pi}{3}$.

18. If we do the problem using washers, the integral we get is $\pi \int_0^1 (1-x)^2 - (1-\sqrt{x})^2 dx$, whereas if we do the problem using shells, the integral we get is $2\pi \int_0^1 y(y-y^2)dy$. Either way the final answer is $\frac{\pi}{6}$.

19. The base is part of a circle, hence $x^2 + y^2 = 4$ and $y = \sqrt{4-x^2}$. Cross sections of the volume above the base are right isosceles triangles so their heights are also y. Their areas are $\frac{1}{2}y \cdot y = \frac{1}{2}(4-x^2)$. Using symmetry we double one half to get Volume $= 2\int_0^2 \frac{1}{2}(4-x^2)\,dx = (4x - \frac{1}{3}x^3)|_0^2 = \frac{16}{3}$.

20. We compute that $f'(x) = \frac{\sec x \tan x}{\sec x} = \tan x$, so the arc length is $\int_0^{\pi/4} \sqrt{1+\tan^2 x}dx = \int_0^{\pi/4} \sec x dx = \ln|\sec x + \tan x|_0^{\pi/4} = \ln|\frac{2}{\sqrt{2}} + 1| - \ln|1+0| = \ln(\sqrt{2}+1)$.

21. Let's try to show that $\int_c^d (mx+b)dx$ is equal to $f(\frac{c+d}{2})(d-c)$. We have $\int_c^d (mx+b)dx = mx^2/2 + bx|_c^d = m(d^2/2 - c^2/2) + b(d-c) = (d-c)(\frac{m}{2}(c+d)+b) = f(\frac{c+d}{2})(d-c)$.

22. The average value of f on $[1,3]$ is given by $\frac{1}{2}\int_1^3 f(x)\,dx$, or, equivalently, $\frac{1}{2}\int_1^3 f(u)\,du$.

    We know $\int_1^9 \frac{f(\sqrt{x})}{\sqrt{x}}\,dx$. Let $u = \sqrt{x}$. Then $du = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx$, so $\frac{1}{\sqrt{x}}\,dx = 2du$. The integral $\int_1^9 \frac{f(\sqrt{x})}{\sqrt{x}}\,dx$ becomes $2\int_1^3 f(u)\,du$. Then $2\int_1^3 f(u)\,du = 2so$ $\int_1^3 f(u)\,du = 1$.

23. The point is that you have to pull the chain up as you go, so that at the bottom the force includes the total weight of the chain, 10 kg, whereas at the top the chain does not contribute at all. Writing the force as a function of a coordinate y ranging from 0 to 10, we get $F(y) = (15 + (10-y))g = (25-y)g$ (here g is the acceleration due to gravity, which is approximately 9.8, but we'll just leave it as g). So the work is $W = \int_0^1 0(25-y)gdy = 250 - \frac{y^2}{2}g|_0^{10} = 200g$ newtons. For a slicker solution, notice that since the force contributed by the chain depends linearly on the height, using the previous problem we know that the average force occurs in the middle – it is $5g$. The work is the distance times the average force, so is $10(15g + 5g = 200g$, the correct answer.

24. If we coordinatize the depth of the water using a coordinate y that runs from -10 to 10, then the amount the water must be pumped as a function of y is $d(y) = 10 - y$, whereas the cross-sectional area at y is $A(y) = \pi(100-y^2)$. Since the general formula for the work done in this manner is $W = \int_a^b \rho g A(y)d(y)dy$, our setup is $\int_{-10}^{10} \rho g \pi(100-y^2)(10-y)dy = \frac{\rho g \pi 40{,}000}{3} = \frac{9{,}810\pi(40{,}000)}{3}$, a heck of a lot of work.

25. The spherical tank is completely symmetrical about the plane $y = 0$: for every point inside the sphere with a positive y-coordinate that must

be pumped less than 10 meters, there is a point with the same negative y-coordinate that must be pumped the corresponding amount more than 10 meters. Therefore, we reason that the final answer must be the same as if all the water were being pumped 10 meters, i.e. the work done must be 10 meters times the total weight of the water, so $W = 10(\frac{4}{3}\pi R^3 1000g) = \frac{4}{3}\pi(10)(1000)(9810g) = \frac{9{,}810\pi(40{,}000)}{3}$, the exact same answer. (This technique of using symmetry to circumvent the calculus is one familiar to anyone who has taken a "physics without calculus" course. Just because you know how to integrate doesn't mean that you should forget to think about the situation geometrically – sometimes "elementary" geometric reasoning leasds to a complete solution.

26. Think of x as being the distance from the destination to the load so $f(x) = load + 0.2x$.

    (a) $W_A = \int_0^{40} F(x)\,dx = \int_0^{40}(12 + 0.2x)\,dx = \ldots = 640$ ft-lbs.

    (b) $W_B = \int_0^{30} F(x)\,dx = \int_0^{30}(16 + 0.2x)\,dx = \ldots = 570$ ft-lbs. Amelia does 70 ft-lbs more work than Beulah.

27. Let h be the distance that a thin horizontal slice of milk is below the rim. (There are other perfectly good ways to set this up!) The rectangular area of that slice will be $(5)(2r)$ where $h^2 + r^2 = 9$. Then

    $W = 64.5 \cdot 5 \int_0^3 \sqrt{9 - h^2} 2h\,dh = 322.5\,\frac{-2}{3}(9 - h^2)^{3/2}|0^3 = 5805$ ft-lbs.

28. (a) Slice from $y = 0$ to $y = 1.8$ parallel to the base. Use a uniform partition. The weight of the ith slice is approximately $\rho(y)\pi r^2 \Delta y$ where $r =$ the $x-$ value on the $y = \ln x$ curve. In other words, $r = e^y$. The weight of the ith slice is approximately $\rho(y)\pi e^{2y}\Delta y$.

    Summing and taking the limit as the number of slices grows without bound gives $\int_0^{1.8} \rho(y)\pi e^{2y}\,dy$.

    (b) Let y be the distance from the bottom of the fountain. Again, slice parallel to the base.We'll need to multiply the weight of a slice by the distance the slice must travel. The work required to lift the ith slice over the rim is approximately $\rho(y)\pi r^2\Delta y \cdot (2 - y)$ where $r =$ the $x-$ value on the $y = \ln x$ curve. The work required to lift the ith slice over the rim is approximately $\rho(y)\pi e^{2y}\Delta y \cdot (2 - y)$ The total work necessary is given by $\int_0^{1.8} \rho(y)\pi e^{2y} \cdot (2 - y)\,dy$.

29. (a) $\int_{-infty}^{\infty} f(t)\,dt = 1$. Since f is zero for x negative, we have $\int_0^{\infty} f(t)\,dt = 1$. Do the integral, take the limit, and solve for N.

    (b) $\int_0^2 f(t)\,dt$. Do the integral to get the answer.