1. (a) $\sum n = 1^\infty |a_n|$ converges, so we know that $\lim_{n\to\infty} a_n = 0$.
   Therefore, for n large enough we know that $|a_n| < 1$. It follows that $(a_n)^2 < |a_n|$ for n large enough. Thus the series converges by direct comparison to $\sum_{n=1}^\infty |a_n|$.

   (b) We'll show $\sum_{n=1}^\infty |(a_n)^3|$ converges. This will mean that $\sum_{n=1}^\infty (a_n)^3$ converges absolutely and therefore converges.
   As argued in part (a), for n large enough we know that $|a_n| < 1$. It follows that $|(a_n)^3| < |a_n|$ for n large enough.Therefore $\sum_{n=1}^\infty |(a_n)^3|$ converges by direct comparison to $\sum_{n=1}^\infty |a_n|$.

   (c) This diverges by the Nth term test for Divergence. We know that $\lim_{n\to\infty}(a_n)^2 = 0$. (You can argue this in several ways. One of these is using part (a).) Therefore $\lim_{n\to\infty} \frac{1}{(a_n)^2} = \infty \neq 0$.

   (d) We'll show $\sum_{n=1}^\infty |(\frac{2}{3})^n(a_n)|$ converges. This will mean that $\sum_{n=1}^\infty(\frac{2}{3})^n(a_n)$ converges absolutely and therefore converges.
   $(\frac{2}{3})^n < 1$, so $|(\frac{2}{3})^n(a_n)| < |a_n|$. Therefore $\sum_{n=1}^\infty |(\frac{2}{3})^n(a_n)|$ converges by direct comparison to $\sum_{n=1}^\infty |a_n|$.

2. (a) Insufficient information. If for instance $a_n = \frac{1}{n^2}$ then the series will converge but if $a_n = \frac{1}{n2}$ then the series will diverge.

   (b) Insufficient information. If for instance $a_n = \frac{1}{n^2}$ then the series will converge. On the other hand, if $a_n = \frac{1}{\ln n}$ then the series $\sum_{n=1}^\infty |\frac{1}{n \ln n}|$ will diverge. You can show this will diverge using the integral test.
   (Note: this problem was harder than I anticipated when I wrote it. We would not expect you to come up with this on your own.)

   (c) Converges. Factor $\frac{10}{3}$ out of the sum.

   (d) Insufficient information. f for instance $a_n = \frac{1}{n^3}$ then the series will converge but if $a_n = \frac{1}{n}$ then the series will diverge.

3. Let $S_n = b_1 + b_2 + b_3 + \ldots + b_n$. It means that $\lim_{n\to\infty} S_n = 11$.

4. (i) The series $\sum_{n=0}^\infty \frac{a_n}{2^n}$ converges. This can be seen as follows. $\sum_{n=0}^\infty a_n$ and $\sum_{n=0}^\infty \frac{a_n}{2^n}$ are both series with positive terms, $\sum_{n=0}^\infty a_n$ converges, and for all $n \geq 0$, $\frac{a_n}{2^n} \leq a_n$. By the comparison test for convergence, we conclude that $\sum_{n=0}^\infty \frac{a_n}{2^n}$ also converges.

   (ii) The series $\sum_{n=0}^\infty \frac{(-1)^n}{a_n}$ diverges. Since $\sum_{n=0}^\infty a_n$ converges, we must have $\lim_{n\to\infty} a_n = 0$, therefore $\lim_{n\to\infty} \frac{1}{a_n} = \infty$, taking into account the fact that all a_ns are positive. Thus the sequence $\frac{(-1)^n}{a_n}$ diverges, and by the divergence test the series $\sum_{n=0}^\infty \frac{(-1)^n}{a_n}$ diverges.

   (iii) The series $\sum_{n=0}^\infty(\sin n)a_n$ converges, in fact it converges absolutely. Indeed, since $|\sin n| \leq 1$, we have that for all $n \geq 0$, $|\sin n|a_n \leq a_n$, and $\sum_{n=0}^\infty |\sin n|a_n$ converges by comparison with $\sum_{n=0}^\infty a_n$. This means that $\sum_{n=0}^\infty(\sin n)a_n$ also converges, because an absolutely convergent series is also convergent.

   (iv) It cannot be determined whether $\sum_{n=0}^\infty \left(\frac{5}{4}\right)^n a_n$ converges or diverges. If $\sum_{n=0}^\infty a_n = \sum_{n=0}^\infty \left(\frac{4}{5}\right)^n$ $(a_n = \left(\frac{4}{5}\right)^n)$ for instance, then $\sum_{n=0}^\infty \left(\frac{5}{4}\right)^n a_n = \sum_{n=0}^\infty 1$ is divergent. If $\sum_{n=0}^\infty a_n = \sum_{n=0}^\infty \left(\frac{2}{5}\right)^n$ $(a_n = \left(\frac{2}{5}\right)^n)$, then $\sum_{n=0}^\infty \left(\frac{5}{4}\right)^n a_n = \sum_{n=0}^\infty \left(\frac{1}{2}\right)^n$ is convergent.

5. First we verify the statements in the hint:
   (i) The derivative of an even function is odd.
   If f is even then $f(x) = f(-x)$.
   Take the derivative of each side.$\frac{d}{dx}f(x) = \frac{d}{dx}f(-x)$. This gives $f'(x) = -f'(-x)$. f' is an odd function.
   (ii) The derivative of an odd function is even.
   If f is even then $f(x) = -f(-x)$.
   Take the derivative of each side.$\frac{d}{dx}f(x) = \frac{d}{dx} - f(-x)$. This gives $f'(x) = f'(-x)$. f' is an even function.
   Note that if g is odd then $g(0) = 0$.

The Taylor series generated by f about $x = 0$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

(a) If f is even then f' is odd and f'' is even, f''' is odd, etc. In other words, $f^{(n)}$ is even if n is even and odd if n is odd. But then if n is odd $f^{(n)}$ is odd and $f^{(n)}(0) = 0$, so the Taylor series centered at $x = 0$ has only even powers of x.

(b) If f is odd then f' is even and f'' is odd, f''' is even, etc. In other words, $f^{(n)}$ is even if n is odd and odd if n is even. But then if n is even $f^{(n)}$ is odd and $f^{(n)}(0) = 0$, so the Taylor series centered at $x = 0$ has only odd powers of x.

6. $\sin u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!}$ for all u.

   Let $u = x^2$.

   Then $\sin x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$ for all x.

   Divide by x. Then $\frac{\sin x^2}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n+1)!}$ for all x.

7. First let's do 7b (which should have been its own problem but got stuck in the middle of this in a TeX error.)

   (7b) The Taylor series generated by f about $x = \frac{\pi}{4}$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{4})}{n!} (x - \frac{\pi}{4})^n.$$

   Take the derivatives of $\sin x$ and evaluate them at $\pi/4$. We know that $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$. We get $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \frac{\pi}{4}) - \frac{1}{2!\sqrt{2}}(x - \frac{\pi}{4})^2 - \frac{1}{3!\sqrt{2}}(x - \frac{\pi}{4})^3 + \frac{1}{4!\sqrt{2}}(x - \frac{\pi}{4})^4 + \frac{1}{5!\sqrt{2}}(x - \frac{\pi}{4})^5 + + \frac{1}{6!\sqrt{2}}(x - \frac{\pi}{4})^6 - + \frac{1}{7!\sqrt{2}}(x - \frac{\pi}{4})^7$

7.

   (a) We know $\cos u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!}$ for all u.

   Let $u = x^3$.

   $\cos x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}$ for all x.

   $5x \cos x^3 = 5 \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!}$ for all x.

   (c) I'll write this in summation notation, but it's actually easier to just write out the first 4 terms to answer this part of the problem.

$$\int_0^{0.2} 5x \cos x^3\, dx = 5 \int_0^{0.2} \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!} \right] dx = 5 \sum_{n=0}^{\infty} \left[\int_0^{0.2} (-1)^n \frac{x^{6n+1}}{(2n)!}\, dx \right] = 5 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \int_0^{0.2} x^{6n+1}\, dx$$

$$\int_0^{0.2} 5x \cos x^3\, dx = 5 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \frac{x^{6n+2}}{6n+2} \Big|_0^{0.2} = 5 \sum_{n=0}^{\infty} (-1)^n \frac{(0.2)^{6n+2}}{(2n)!(6n+2)}$$

$$\int_0^{0.2} 5x \cos x^3\, dx = \frac{5(0.2)^2}{2} - \frac{5(0.2)^8}{2 \cdot 8} + \frac{5(0.2)^{14}}{4! \cdot 14} - \frac{5(0.2)^{20}}{6! \cdot 20} + \dots$$

   (You could simplify this more, but I won't.)

   (d) We can use the alternating series error estimate because the series is alternating in sign, the terms are decreasing in magnitude, and the terms tend towards zero. The third term is less than 10^{-6}, so only the first two non-zero terms of the series are necessary.

8. (a) To find the Taylor series of $f(x)$ at $x = 10$ one can compute the following:

$$f(10) + f'(10)(x-10) + \frac{f''(10)}{2!}(x-10)^2 + \frac{f'''(10)}{3!}(x-10)^3 + \ldots + \frac{f^{(n)}(10)}{n!}(x-10)^n + \ldots$$

$f(x) = \frac{1}{1-x}$	$f(10) = \frac{1}{(-9)}$
$f'(x) = (1-x)^{-2}$	$f'(10) = \frac{1}{(-9)^2}$
$f''(x) = 2(1-x)^{-3}$	$f''(10) = 2!\frac{1}{(-9)^3}$
$f'''(x) = (3)(2)(1-x)^{-4}$	$f'''(10) = 3!\frac{1}{(-9)^4}$
$f^{(4)}(x) = 4!(1-x)^{-5}$	$f(4)(10) = 4!\frac{1}{(-9)^5}$
...	
$f^n(x) = n!(1-x)^{-(n+1)}$	$f^{(n)}(10) = n!\frac{1}{(-9)^{(n+1)}}$

This gives us

$$-\frac{1}{9} + \frac{1}{9^2}(x-10) - \frac{2!}{9^3 2!}(x-10)^2 + \frac{3!}{9^4 3!}(x-10)^3 + \ldots + (-1)^{n+1}\frac{n!}{9^{(n+1)}n!}(x-10)^n + \ldots$$

or

$$-\frac{1}{9} + \frac{(x-10)}{9^2} - \frac{(x-10)^2}{9^3} + \frac{(x-10)^3}{9^4} + \ldots + (-1)^{n+1}\frac{(x-10)^n}{9^{(n+1)}} + \ldots$$

An alternative method is the following: To find the Taylor series of $f(x)$ at $x = a$ is the same as to find the Maclaurin series of $f(a+t)$ at $t = 0$ and then plug $x = t - a$. In our case, $f(x) = \frac{1}{1-x}$ and $a = 10$ so that

$$\begin{aligned} f(10+t) = 1/(1-(10+t)) &= -1/(t+9) = (-1/9)(1/(1+t/9)) \\ &= -1/9(1 - t/9 + (t/9)^2 - (t/9)^3 + \cdots) \\ &= \sum_{n=0}^{\infty}(-1)^{n+1}t^n/9^{n+1}. \end{aligned}$$

Here $t = x - 10$.

(b) The geometric series $-1/9\sum_{n=0}^{\infty}(-1)^n\frac{(x-10)^n}{9^n}$ (with $a = -1/9$ and $r = \frac{(x-10)}{-9}$ converges to $\frac{-1/9}{1+\frac{x-10}{9}}$ for $|\frac{x-10}{-9}| < 1$ Therefore, for $|x-10| < 9$ the series converges to $f(x)$. The radius of convergence is thus 9.

(c) Since $|17-10| < 9$, the series converges to $f(17)$.

(d) Since $|1/2-10| > 9$, the series diverges.

9. One can use repeated differentiation to compute the values of f and of its derivatives at 5, or one can simply observe that the Taylor series about $x = 5$ for a polynomial in $(x-5)$ is the polynomial itself. Therefore by identifying the coefficients of the Taylor series to that of the given polynomial one gets:

$$f(5) = \sqrt{3} \quad f''(5) = 0 \quad \frac{f'''(5)}{3!} = 12 \quad \frac{f^{(6)}(5)}{6!} = 17.$$

10. The first few derivatives of $f(x) = \cos x$ and their values at $-\pi/2$ are:

$$\begin{aligned} f(x) = \cos x &\quad f(-\pi/2) = 0 \\ f'(x) = -\sin x &\quad f'(-\pi/2) = 1 \\ f''(x) = -\cos x &\quad f''(-\pi/2) = \\ f'''(x) = \sin x &\quad f'''(-\pi/2) = -1 \\ f^{(4)}(x) = \cos x &\quad f^{(4)}(-\pi/2) = 0 \end{aligned}$$

After that the pattern repeats itself. Using the formula giving the Taylor series in terms of the values of the derivatives we obtain:

$$\begin{aligned}\cos(x) &= (x+\pi/2) - \frac{(x+\pi/2)^3}{3!} + \frac{(x+\pi/2)^5}{5!} - \frac{(x+\pi/2)^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{(x+\pi/2)^{2n+1}}{(2n+1)!}.\end{aligned}$$

11. (a) The function $\cosh x$ is even, and it goes to infinity exponentially fast as $x \to \pm\infty$. So the graph looks like a parabola going to infinity exponentially.
The function $\sinh x$ is odd, and it goes to ∞ exponentially fast as $x \to \infty$, and to $-\infty$ as $x \to -\infty$. Thus the graph looks like the graph of x^3, except that it goes much faster to infinity.
(b)

$$\frac{d\cosh x}{dx} = \sinh x \qquad \frac{d\sinh x}{dx} = \cosh x$$

(c) We know the Taylor series for e^x, and also for e^{-x} (by substituting $-x$ for x in the Taylor series for e^x):

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \end{aligned}$$

Subtracting the second series from the first term by term (this is allowed since both series converge absolutely for all x) and dividing by two we get:

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

(d) Taking the derivative of the series above term by term, we obtain:

$$\cosh x = \frac{d\sinh x}{dx} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

12 i. $\ln(\frac{1+x}{1-x}) = \ln(1+x) - \ln(1-x)$. So in order to find Taylor series for $\ln\frac{1+x}{1-x}$ we have to find Taylor series for $\ln(1+x)$ and $\ln(1-x)$.
$\ln(1-x) = -\int \frac{dx}{1-x}$ and $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. So

$$\ln(1-x) = -\int \sum_{n=0}^{\infty} x^n dx = -\sum_{n=0}^{\infty} \int x^n dx = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C.$$

Since for $x = 0$ we have $\ln(1-x) = \ln 1 = 0$, $\quad C = 0$.
Using substitution we can find Taylor series for $\ln(1+x)$:

$$\ln(1+x) = \ln(1-(-x)) = -\sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Finaly, for $\ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$ we have

$$\ln\frac{1+x}{1-x} = -\sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{-(-x)^{n+1} + x^{n+1}}{n+1}.$$

If n is odd $-(-x)^{n+1} + x^{n+1} = 0$, if $n = 2k$ is even $-(-x)^{n+1} + x^{n+1} = 2x^{n+1}$. So in the Taylor series for $\ln\frac{1+x}{1-x}$ we'll have only odd powers of x and the power series will be

$$\ln\frac{1+x}{1-x} = \sum_{k=0}^{\infty} \frac{2x^{2k+1}}{2k+1} = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \ldots.$$

ii. If $x = \frac{1}{3}$ $1 + x = \frac{4}{3}$ and $1 - x = \frac{2}{3}$. So $\frac{1+x}{1-x} = 2$.

iii. Using the first four nonzero terms of series in (a) for $x = 1/3$ we get

$$\ln 2 \approx \frac{2}{3} + \frac{2}{3^3 \cdot 3} + \frac{2}{3^5 \cdot 5} + \frac{2}{3^7 \cdot 7} = \frac{53056}{76545} \approx 0.693134757332.$$

Using the first four terms of the series for $\ln(1+x)$ and $x = 1$ we get

$$\ln 2 \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12} \approx 0.58333$$

Using calculator $\ln 2 \approx 0.69314718056$.

13 i. $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^{n-1}$ is a geometric series with $r = \frac{e}{\pi}$. Since $\frac{e}{\pi} < 1$, the series converges.

ii. Series $\sum_{n=1}^{\infty} \frac{2n^2+1}{3n^2+n}$ is divergent by N-th tern test:

$$\lim_{n\to\infty} \frac{2n^2+1}{3n^2+n} = \lim_{n\to\infty} \frac{2+\frac{1}{n^2}}{3+\frac{1}{n}} = \frac{2}{3} \neq 0.$$

iii. Series $\sum_{n=1}^{\infty} \frac{(-1)^n \pi}{\sqrt{3n}}$ is convergent by AST: signs alternates and absolute value of terms is decreasing and goes to 0.

iv. Series $\sum_{n=1}^{\infty} \frac{3^n}{4^n n!}$ converges. To see this we can apply either comparison or ratio test.

A. For any n $\frac{3^n}{4^n n!} \leq \frac{3^n}{4^n}$. Series $\sum_{n=1}^{\infty} \frac{3^n}{4^n}$ is geometric with $r = \frac{3}{4}$ and hence converges. By comparison test series we started with also converges.

B. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^{n+1}}{4^{n+1}(n+1)!} \frac{4^n n!}{3^n} = \lim_{n\to\infty} \frac{3}{4(n+1)} = 0 \leq 1$. Series is convergent by Ratio test.

v. Series $\sum_{n=1}^{\infty} \frac{\sqrt{3}}{n(n-4)}$ converges by limit comparison test. Let's compare this series to $\sum \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\frac{\sqrt{3}}{n(n-4)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sqrt{3}n^2}{n(n-4)} = \lim_{n\to\infty} \frac{\sqrt{3}}{1-\frac{4}{n}} = \sqrt{3}$$

By limit comparison test this series converges, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

vi. Terms of the series $\sum_{n=1}^{\infty} \frac{\cos n}{(1.1)^n}$ change signs. Let's show, that this series is absolutely convergent. For any n $\left|\frac{\cos n}{(1.1)^n}\right| \leq \frac{1}{(1.1)^n}$. $\sum_{n=1}^{\infty} \frac{1}{(1.1)^n}$ is geometric series with $r = \frac{1}{1.1}$ and hence converges. By comparison test series $\sum_{n=1}^{\infty} \left|\frac{\cos n}{(1.1)^n}\right|$ converges. This means that series we started with is absolutely convergent and hence convergent.

14. For any real number p, the alternating p-series is the infinite series

$$\Sigma_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots$$

a) Show that the alternating p-series diverges for $p \leq 0$.
Solution: There are two cases to consider.
$p = 0$: Then the general term is $(-1)^{n-1}$, so the limit of the nth term does not exist – the sequence alternates between 1 and -1. By the n-th term test, the series diverges.
$p < 0$: If p is negative, $\frac{1}{n^p} = n^{-p}$ and $-p$ is positive. The limit of a positive power of n as n approaches infinity is infinity, so the nth term goes to infinity in absolute value. Therefore the series again diverges by the nth term test.

b) Show that the alternating p-series is convergent but not absolutely convergent if $0 < p \leq 1$.
Solution: To see that the series is convergent, notice that 1) it is alternating, 2) the terms are decreasing: $n^p < (n+1)^p$ so $\frac{1}{n^p} > \frac{1}{(n+1)^p}$, and 3) the general term goes to zero: $n^p \to \infty$, so implies $\frac{1}{n^p} \to 0$. Therefore the alternating series test applies to show the convergence of the series. As for the lack of absolute convergence: when we take absolute values we get the conventional p-series, which we learned

(using the integral test) diverges for $p \leq 1$.

c) Show that the alternating p-series is absolutely convergent for $p > 1$.
Solution: As above, absolute convergence means that the series must converge upon taking absolute values, but upon taking absolute values we again get the usual p-series. Since $p > 1$ it converges, which gives absolute convergence of the alternating p-series.
15.a) Use the Ratio Test: Look at $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. Using lhospitals rule, establish the following limit.

$$\lim_{n\to\infty} \left| \frac{n(\ln(n+1)))x^{n+1}}{(n+1)(\ln(n+2))x^n} \right| = |x|$$

then conclude using the ratio test that the radius of convergence of the power series is 1.

b) When $x = -1$ use the alternating series test to prove that the corresponding series converges. for $x = 1$ proceed like this; use the substitution $u = \ln(1+x)$ to show that the integral $\int_0^\infty \frac{1}{(1+x)\ln(1+x)} dx$ diverges. By direct comparison, conclude that $\int_0^\infty \frac{1}{x\ln(1+x)} dx$ also diverges. Now by the integral test, conclude that the series with $x = 1$ diverges. Finally, conclude that the interval of convergence is $[-1, 1)$

16. Use the Ratio Test: Look at $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. Establish the following limit.

$$\lim_{n\to\infty} \left| \frac{n(x-3)^{n+1}}{(n+1)(x-3)^{n+1}} \right| = |x-3|$$

then conclude using the ratio test that the radius of convergence of the power series is 1. When $x = 4$ conclude that the series is the harmonic series and thus diverges. When $x = 2$ use the alternating series test to conclude the series converges. Thus the interval of convergence is $[2, 4)$

17.Use the Ratio Test: Look at $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. Establish the following limit.

$$\lim_{n\to\infty} \left| \frac{10^{n+1}(2x-5)^{n+1}}{10^{n+2}(2x-5)^n} \right| = \frac{|2x-5|}{10}$$

now note that this limit is less than 1 precisely when $|x - 2.5| < 5$. Thus the radius of convergence is 5. When $x = 7.5$ the series is just

$$\sum \frac{1}{10}$$

which clearly diverges by the nth term test for divergence. When $x = -2.5$ the series is just

$$\sum (-1)^n \frac{1}{10}$$

which also diverges by the nth term test for divergence. THus the interval of convergence is $(-2.5, 7.5)$

18. There are of course (infinitely) many power series of given radius of convergence. we will give one example in each case. For convenience we can consider a power series centered at 0, given by $\sum_{n=0}^\infty a_n x^n$.
    i. $R = 0$
    If

    $$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| > 1$$

    for all $x \neq 0$, then by ratio test the series diverges for all $x \neq 0$, and the radius of convergence is 0. If we take $a_n = n!$, so that $a_{n+1}/a_n = n+1$, then

    $$\lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = \lim_{n\to\infty} |(n+1)x| = \infty, \quad \text{for all x} \neq 0.$$

    In particular the series $\sum_{n=0}^\infty n!x^n$ has radius of convergence 0.

ii. $R = \infty$

This time we should look for a_n's that make this limit exist and

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| < 1,$$

for all x, then the series converges for all x, and so have radius of convergence ∞. If we take $a_n = 1/n!$, so that $a_{n+1}/a_n = 1/(n+1)$, then

$$\lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1, \quad \text{for all x.}$$

Therefore the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ has radius of convergence ∞. (Does this series look familiar?)

iii. $R = 5$

Now if $a_{n+1}/a_n = 1/5$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \frac{|x|}{5},$$

So by ratio test, the series converges if $\frac{|x|}{5} < 1$, and diverges if $\frac{|x|}{5} > 1$. Therefore the radius of convergence of this series is 5. Taking $a_n = \frac{1}{5^n}$ would satisfy $a_{n+1}/a_n = 1/5$, and these can be the coefficients of our series.

19. The center of the power series should be the the center of its interval of convergence. So a power series with interval of convergence $(0, 12)$should be centered at 6, and has radius of convergence 6. So the power series should look like $\sum_{n=0}^{\infty} a_n(x-6)^n$. And similar to the above question, if we take $a_n = \frac{1}{6^n}$, then the series $\sum_{n=0}^{\infty}\frac{(x-6)^n}{6^n}$ has radius of convergence 6. And this particular series doesn't converge for the endpoints $x = 0$, (where we get $\sum_{n=0}^{\infty} 1$), and $x = 12$ (where we get $\sum_{n=0}^{\infty}(-1)^n$).

20. Suppose at the end of the n-th day, there're s_n bees in the beehive, out of which a_n bees are newcomers. Obviously $s_n = a_1 + \ldots + a_n$. At the end of the first day, the beehive has only 1 bee, which is a newcomer, i.e. $a_1 = 1$, $s_1 = 1$. Since every newcomer brings home 3 new roommates the next day, we have $a_{n+1} = 3a_n$. Hence by induction $a_n = 3^n$. Then $s_n = 1 + 3 + \ldots + 3^n = \frac{1-3^{n+1}}{1-3}$. We need to solve n for $\frac{1-3^{n+1}}{1-3} = 364$, and the solution is $n = 6$.

21. Draw a picture for yourself.

i. area of $s_0 = 2^2 = 4$

area of $s_1 = 2$; area of $s_2 = 1$; area of $s_3 = 1/2$

ii. sum of the areas of all the squares $= 4 + 2 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots = \frac{4}{1-\frac{1}{2}} = 8$

(This is a geometric series.)

22. $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all x.

Therefore, $e^{-1} = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$.

This is an alternating series with the terms decreasing in magnitude and tending towards zero, so we can use the alternating series error estimate. We look for the first term whose absolute value is less than 10^{-3}.

$n = 7$ is the smallest n for which $\frac{1}{n!} < \frac{1}{10^3}$. Therefore we need to use the approximation $e^{-1} \approx \displaystyle\sum_{n=0}^{6}\frac{(-1)^n}{n!}$ - we go up to the $\frac{1}{6!}$ term. This ends up being 7 non-zero terms.

In other words, $1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}$ is an approximation of $\frac{1}{e}$ with error less than 10^{-3}.

23. This is the 4th degree Taylor polynomial for $\cos x$ about $x = 0$. We know the Taylor series generated by $\cos x$ about $x = 0$ converges to $\cos x$ for all x. For $|x| < 0.5$ this is an alternating series with the terms decreasing in magnitude and tending towards zero, so we can use the alternating series error estimate. The first unused term has magnitude $\frac{|x^6|}{6!}$. For $|x| < 0.5$ we know $\frac{|x^6|}{6!} < \frac{|(\frac{1}{2})^6|}{6!} = \frac{1}{2^6\cdot 6!} = \frac{1}{32\cdot 720} = \frac{1}{23040}$

24. As this is an alternating series, we can estimate the error by using the alternating series error estimate. This is that the size of the error is less than the first term that is not included in the approximation. In this case, we want the error to be less than than .01.

    Thus we would like to find an odd number n whose inverse is less than .01. The first one is 101. Thus our last term can be 1/99, which means that we have 50 terms. This can be seen by noting that the last term of the partial sum with n terms will be plus or minus $1/(1+2(n-1))$.

25. a. The easiest way to do this is to use the binomial series expansion with $k = 1/2$. The result is

$$1 + \frac{x}{2} - \frac{x^2}{8}$$

b. When $x = .2$ this series is alternating if we add the first two terms to make a single term. Thus, we can use the alternating series error estimate. The first term not included is 3/16, which our upper bound for the error.

c. When $x = -.3$ this series is no longer alternating. Thus we have to use Taylor with remainder and Taylor's inequality i.e. $f(x) = T_n(x) + R_n(x)$ where

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

and M is a constant that is greater than or equal to $|f^{(n+1)}(c)|$ for all c between x and a. Here, we have $n = 2$ (because we have three terms), so we have to find an upper bound for $|f^{(3)}|$ between 0 and 1. Taking derivatives,

$$|f^{(3)}(x)| = \frac{1}{4|1+x|^{3/2}}$$

which obtains its maximum at $x = 0$. Thus we can take M equal to 1/4, and substituting our M as well as $a = 0$, we get

$$|R_2(x)| \leq \frac{|x|^3}{4 \cdot 3!} = \frac{|x|^3}{24}.$$

so

$$|R_2(-.3)| \leq \frac{(.3)^3}{24}.$$