Math 1B: Second Midterm

Harvard University
Wednesday, 1 December 1999, 7:30 – 9:30 pm
Solutions

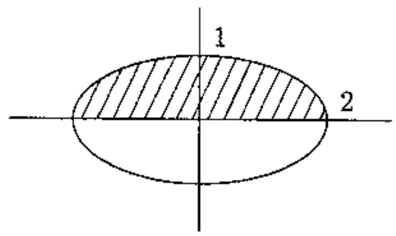
Note: $\ln x = \log x$.

1. (10 points) Using integration, find the surface area of a circular cone of height 3 and radius 4, ignoring the circular base.

   Answer: The sides of the cone are obtained by rotating the line $y = (4/3)x$ about the x-axis, from $x = 0$ to $x = 3$, giving area:

$$\begin{aligned} A &= \int_0^3 2\pi y\sqrt{1+(y')^2}\,dx = \int_0^3 \frac{8\pi}{3}x\sqrt{1+\left(\frac{4}{3}\right)^2}\,dx \\ &= \frac{8\pi}{3}\sqrt{\frac{5^2}{3^2}}\,\frac{x^2}{2}\Big|_0^3 \\ &= 4\pi\cdot 5 = 20\pi. \end{aligned}$$

2. (15 points) Find the volume of the solid obtained by revolving, around the x-axis, the shaded region above the x-axis and inside the ellipse $x^2/4 + y^2 = 1$.



   Answer:

$$\begin{aligned} V &= \int_{-2}^2 \pi y^2\,dx = \int_{-2}^2 \pi(1 - x^2/4)\,dx \\ &= \pi(x - x^3/12)|_{-2}^2 = 2\pi(2 - 8/12) = 8\pi/3. \end{aligned}$$

3. (10 points) Compute the following integrals.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$$

Answer: Using the substitution $u = \sqrt{x}$, $du = dx/(2\sqrt{x})$, we obtain:

$$\begin{aligned} \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx &= \int 2e^u\,du \\ &= 2e^u = 2e^{\sqrt{x}} + C. \end{aligned}$$

$$\int \frac{x^2}{x^2+x-2}\,dx$$

Answer: First, make the fraction *proper:*

$$\frac{x^2}{x^2+x-2} = \frac{x^2+x-2-x+2}{x^2+x-2} = 1 + \frac{-x+2}{x^2+x-2}.$$

(You can also do this step by long division.) Next, factor the denominator as $x^2 + x - 2 = (x+2)(x-1)$ and use partial fractions:

$$\frac{-x+2}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1} = \frac{(A+B)x + (2B-A)}{(x+2)(x-1)} \quad \Longrightarrow$$

$$A + B = -1, \quad 2B - A = 2.$$

Adding these equations, we find $3B = 1$, so $B = 1/3$ and $A = -4/3$. Thus:

$$\begin{aligned} \int \frac{x^2}{x^2+x-2}\,dx &= \int 1 + \frac{-x+2}{x^2+x-2}\,dx \\ &= \int 1 - \frac{4}{3(x+2)} + \frac{1}{3(x-1)}\,dx \\ &= x - \frac{4}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C. \end{aligned}$$

4. (10 points) Compute the following integral.

$$\int \frac{dx}{x^{1/2}(1+x^{1/4})}$$

Answer: Substitute $u = 1 + x^{1/4}$; then $x = (u-1)^4$, and $dx = 4(u-1)^3\,du$. Then we get:

$$\begin{aligned} \int \frac{dx}{x^{1/2}(1+x^{1/4})} &= \int \frac{4(u-1)^3\,du}{(u-1)^2 u} = \int \frac{4(u-1)\,du}{u} \\ &= 4\int 1 - \frac{1}{u}\,du = 4(u - \ln u) \\ &= 4(1 + x^{1/4} - \ln(1+x^{1/4})) + C \\ &= 4x^{1/4} - 4\ln(1+x^{1/4}) + C'. \end{aligned}$$

(You can also compute dx directly from $u = 1 + x^{1/4}$: then $du = (1/4)x^{-3/4}dx$ and so $dx = 4x^{3/4}du = 4(u-1)^3$.)

5. (15 points) Determine if the improper integrals below converge or diverge. Compute the values of the convergent integrals.

$$\int_0^1 \ln x \, dx$$

Answer: The integral above is improper because $\ln x = \ln x$ blows up at $x = 0$. By integration by parts, $\int \ln x \, dx = x \ln x - x$. So we get:

$$\begin{aligned} \int_0^1 \ln x \, dx &= \lim_{a \to 0+} \int_a^1 \ln x \, dx = \lim_{a \to 0+} x \ln x - x|_a^1 \\ &= -1 + \lim_{a \to 0+} a \ln a - a \\ &= -1 + 0 = -1. \end{aligned}$$

Here we have used L'Hôpital's rule to evaluation the limit:

$$\lim_{a \to 0+} a \ln a = \lim_{a \to 0+} \frac{\ln a}{1/a} = \lim_{a \to 0+} \frac{1/a}{-1/a^2} = \lim_{a \to 0+} -a = 0.$$

$$\int_2^4 \frac{dx}{|x - \pi|}$$

Answer: We have $2 < \pi < 4$, so the integrand blows up at $x = \pi$. Also $|x - \pi| = \pm(x - \pi)$, depending on the sign of $x - \pi$. Taking this into account, we get:

$$\int_2^4 \frac{dx}{|x-\pi|} = \lim_{b \to \pi-} \int_2^b \frac{dx}{\pi - x} + \lim_{a \to \pi+} \int_a^4 \frac{dx}{x - \pi}.$$

Now the first limit is:

$$\lim_{b \to \pi-} \int_2^b \frac{dx}{\pi - x} = \lim_{b \to \pi-} -\ln(\pi - x)|_2^b = \ln(\pi - 2) - \lim_{b \to \pi-} \ln(\pi - b) = \infty,$$

since $\ln x$ diverges at $x = 0$.

Thus the integral **diverges.**

6. (15 points) Determine if the improper integrals below converge or diverge. Compute the values of the convergent integrals.

$$\int_{-1}^8 \frac{dx}{x^{1/3}}$$

Answer: Since $x^{-1/3}$ blows up at $x = 0$, we must break the integral into two parts:

$$\int_{-1}^{8} \frac{dx}{x^{1/3}} = \lim_{b \to 0-} \int_{-1}^{b} \frac{dx}{x^{1/3}} + \lim_{a \to 0+} \int_{a}^{8} \frac{dx}{x^{1/3}}.$$

For the first part we have:

$$\lim_{b \to 0-} \int_{-1}^{b} \frac{dx}{x^{1/3}} = \lim (3/2) x^{2/3} |_{-1}^{b} = \lim (3/2) b^{2/3} - (3/2)(-1)^{2/3} = 0 - (3/2)(1) = -3/2.$$

Similary, for the second part we have:

$$\lim_{a \to 0+} \int_{a}^{8} \frac{dx}{x^{1/3}} = (3/2) 8^{2/3} = (3/2)(4) = 6,$$

and therefore:

$$\int_{-1}^{8} \frac{dx}{x^{1/3}} = 6 - 3/2 = 9/2.$$

$$\int_{-\infty}^{0} t^2 e^t \, dt$$

Answer: Using integration by parts several times, we get:

$$\begin{aligned} \int t^2 e^t \, dt &= \int t^2 d(e^t) = t^2 e^t - \int 2te^t \, dt \\ &= t^2 e^t - \int 2t d(e^t) = t^2 e^t - 2te^t + \int 2e^t \, dt \\ &= e^t (t^2 - 2t + 2). \end{aligned}$$

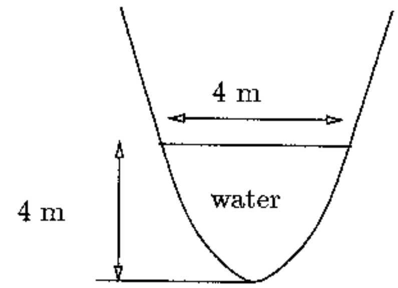
Therefore:

$$\begin{aligned} \int_{-\infty}^{0} t^2 e^t \, dt &= \lim_{a \to -\infty} \int_{a}^{0} t^2 e^t \, dt \\ &= \lim_{a \to -\infty} e^t (t^2 - 2t + 2)|_{a}^{0} \\ &= 2 - \lim_{a \to -\infty} e^a (a^2 - 2a + 2) = 2. \end{aligned}$$

Here we have used L'Hôpital's rule to evaluate the limits; for example,

$$\lim_{a \to -\infty} a^2 e^a = \lim_{a \to +\infty} \frac{a^2}{e^a} = \lim_{a \to +\infty} \frac{2a}{e^a} = \lim_{a \to +\infty} \frac{2}{e^a} = 0.$$

7. (15 points) Compute the total fluid force on the dam described by the equation $y = x^2$ when the water is 4 meters deep. Express your answer as a constant times δ, where δ is the density of water.



Answer: At a given height y, the width of the dam is $w(y) = 2x = 2\sqrt{y}$, and the depth of the water is $4 - y$, so the pressure is $\delta(4-y)$. Thus the total force is:

$$\begin{aligned}
F &= \int_0^4 \delta(4-y)2\sqrt{y}\,dy \\
&= 2\delta\int_0^4 4y^{1/2} - y^{3/2}\,dy = (16\delta/3)y^{3/2} - (4\delta/5)y^{5/2}|_0^4 \\
&= (16\delta/3)8 - (4\delta/5)32 = 128\delta(1/3 - 1/5) \\
&= (256/15)\delta.
\end{aligned}$$

8. (10 points) A leaky pail holding 20 lbs. of water is raised from the bottom of an 18 foot well. By the time the pail reaches the top, it is only half full (it contains just 10 lbs. of water). Assuming the water leaks out at a constant rate, how much work is done? (Ignore the weight of the pail and rope.)

Answer: The weight of the pail decreases by 10 lbs over 18 ft, so the rate of decrease is 10/18 (lbs/ft.) Thus the weight of the pail after is has been raised height x is $F(x) = 20 - 10x/18$, and so the total work done is:

$$\begin{aligned}
W &= \int_0^{18} F(x)\,dx = \int_0^{18} (20 - 10x/18)\,dx \\
&= 20x - 5x^2/18|_0^{18} = 360 - 90 = 270 \text{ ft-lbs.}
\end{aligned}$$