16. $\frac{dx}{dt} = x - x^2 - axy$

$\frac{dy}{dt} = y - y^2 - axy$

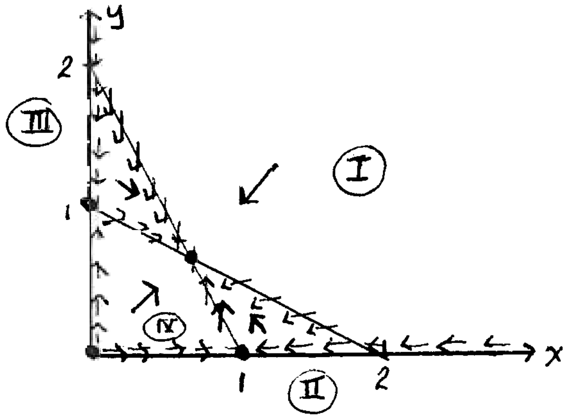
The two species are competitors. Each, in the absence of the other, grows logistically. There's a carrying capacity of 1 for either alone.

$\boxed{\text{PHASE PLANE ANALYSIS WITH } a = \tfrac{1}{2}}$

$\frac{dx}{dt} = x(1 - x - \frac{1}{2}y)$ x-null clines: $x=0$ <u>or</u> $1-x-\frac{1}{2}y=0$ i.e. $y = 2-2x$ vertical slope marks

$\frac{dy}{dt} = y(1-y-\frac{1}{2}x)$ y-null clines: $y=0$ <u>or</u> $1-y-\frac{1}{2}x=0$, i.e. $y = 1-\frac{1}{2}x$ horizontal slope lines

The realistic modeling constraints restrict us to the 1st quadrant: $x \geq 0$, $y \geq 0$.



Label regions in the 1st quadrant I, II, III, and IV as indicated

Equilibrium pts: where x-null clines & y-null clines intersect: $(0,0)$ $(1,0)$ $(0,1)$ and $(\frac{2}{3}, \frac{2}{3})$

<u>Interpreting the phase-plane analysis:</u>

- suppose $y_0 = 0$ (we're on the x-axis). Then $y=0$ for all t. If $x_0 > 1$ then x decreases with t and approaches 1.
  If $x_0 < 1$ and $x_0 > 0$ then x increases with t and approaches 1.
- Similarly, if $x_0 = 0$ then $x=0$ for all t.
  If $y_0 > 1$ then y decreases with t & $y \to 1^+$
  If $0 < y_0 < 1$, then y increases with t & $y \to 1^-$
  (The x and y-axes - being trajectories, can't be crossed!)

- Suppose (x_0, y_0) is in region I.
  The trajectory heads down & left throughout the region.
  Therefore $(x(t), y(t))$ can tend towards $(\frac{2}{3}, \frac{2}{3})$ or will enter regions II or III. See analysis below for what happens next.
- Suppose (x_0, y_0) is in region II.
  The trajectory heads down and right throughout the region.
  The orientation of the slope marks on the null clines bounding region II indicate that the trajectory can't leave region II once it enters it. $(x(t), y(t))$ approaches $(\frac{2}{3}, \frac{2}{3})$.
- Suppose (x_0, y_0) is in region III.
  The trajectory heads up and left throughout the region and can't leave the region. $(x(t), y(t))$ approaches $(\frac{2}{3}, \frac{2}{3})$.
- Suppose (x_0, y_0) is in region IV.
  The trajectory heads up and right throughout the region.
  Therefore $(x(t), y(t))$ heads for $(\frac{2}{3}, \frac{2}{3})$ or enters region II or region III and from there heads to $(\frac{2}{3}, \frac{2}{3})$.

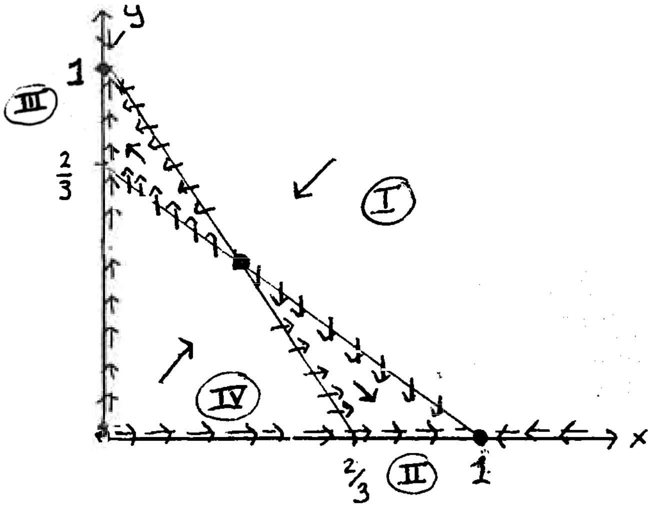
So, if $x_0 > 0$ and $y_0 > 0$, the system will tend towards the equilibrium $(\frac{2}{3}, \frac{2}{3})$.

If $\begin{cases} y_0 = 0 \\ x_0 > 0 \end{cases}$ $x(t) \to 1$. If $\begin{cases} x_0 = 0 \\ y_0 > 0 \end{cases}$ $y(t) \to 1$. If $\begin{matrix} x_0 = 0 \\ y_0 = 0 \end{matrix}$ then $\begin{matrix} x(t) = 0 \\ y(t) = 0 \end{matrix}$

PHASE PLANE ANALYSIS WITH $a = \frac{3}{2}$

$\frac{dx}{dt} = x(1-x-\frac{3}{2}y)$ x-null clines: $x=0$ <u>or</u> $1-x-\frac{3}{2}y=0$ i.e. $y=\frac{2}{3}-\frac{2}{3}x$ vertical slope marks

$\frac{dy}{dt} = y(1-y-\frac{3}{2}x)$ y-null clines: $y=0$ <u>or</u> $1-y-\frac{3}{2}x=0$ i.e $y= -\frac{3}{2}x$ horizontal slope marks



Equilibrium points: where x-null clines and y null clines intersect
$(1,0), (0,1), (0,0)$ and $(\frac{2}{5}, \frac{2}{5})$

<u>Interpreting the phase-plane analysis</u>

- Suppose $y_0=0$ (we're on the x-axis). Then $y(t)=0$ and $x(t)\to 1$ (decreasing if $x_0>1$ and increasing if $x_0<1$)
- Suppose $x_0=0$. Then $x(t)=0$ and $y(t)\to 1$.

- Suppose (x_0, y_0) is in region I.
  The trajectory heads down and left throughout the region
  $(x(t), y(t))$ either approaches $(\frac{2}{5}, \frac{2}{5})$ or will enter regions II or III. See analysis below.
- Suppose (x_0, y_0) is in region III.
  The trajectory heads up and left throughout the region and can't leave the region.
  $(x(t), y(t))$ approaches $(0,1)$.
- Suppose (x_0, y_0) is in region II.
  The trajectory heads down and right throughout the region & can't leave the region. $(x(t), (y(t))$ approaches $(1,0)$.
- Suppose (x_0, y_0) is in region IV,
  the trajectory heads up and right throughout the region.
  $(x(t), y(t))$ either approaches $(\frac{2}{5}, \frac{2}{5})$ or will enter region II and go towards $(1,0)$ or will enter region III and tend towards $(0,1)$.

Compare & contrast: For the case that there is either $x=0$ or $y=0$ the two systems behave the same.

For $x_0>0$ and $y_0>0$ the system with $a=\frac{1}{2}$ tends towards the equilibrium $(\frac{2}{3}, \frac{2}{3})$.

For $x_0>0$ and $y_0>0$ the system with $a=\frac{3}{2}$ tends towards $(1,0)$ or $(0,1)$ or $(\frac{2}{5}, \frac{2}{5})$.

In fact, if $x_0 \neq y_0$ then the trajectory tends to $x=1$ $y=0$ if $x_0>y_0$ and to $x=0$, $y=1$ if $x_0<y_0$. In other words - one "species" will win out and the other will become extinct - in accordance with Darwin's principle of Competitive exclusion.

it is <u>not</u> necessary to say this to do the problem - but it's true. (You can argue this using symmetry & uniqueness.)