Integration Handout A – Solutions

24. (a) $\int \frac{x}{\sqrt{4+x^2}}\,dx$ let $u = 4+x^2 \Rightarrow du = 2x\,dx$

$$= \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\left[2u^{1/2}\right] = \boxed{\sqrt{4+x^2} + C}$$

(b) $\int_0^1 \sqrt{4-t^2}\,dt$ let $t = 2\sin\theta \Rightarrow dt = 2\cos\theta\,d\theta$

$$= \int_0^{\pi/6} \sqrt{4(1-\sin^2\theta)}\cdot 2\cos\theta\,d\theta = 2\int_0^{\pi/6} 2\cos\theta\cdot\cos\theta\,d\theta$$

$$= 4\int_0^{\pi/6}\cos^2\theta\,d\theta = 4\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_0^{\pi/6}\right] = 4\left[\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right] = \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}$$

(c) $\int_0^1 x^3\sqrt{4-x^2}\,dx$ let $x = 2\sin\theta$ $dx = 2\cos\theta\,d\theta$

$$= \int_0^{\pi/6} 8\sin^3\theta\,\sqrt{4(1-\sin^2\theta)}\cdot 2\cos\theta\,d\theta = 32\int_0^{\pi/6}\sin^3\theta\,\underbrace{\cos^2\theta}_{(1-\sin^2\theta)}\,d\theta$$

$$= 32\int_0^{\pi/6}\sin^3\theta\,d\theta - 32\int_0^{\pi/6}\sin^5\theta\,d\theta$$

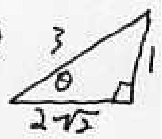
$$= 32\left[-\frac{2\cos\theta}{3} - \frac{\cos\theta\sin^2\theta}{3}\Big|_0^{\pi/6}\right] - 32\left[\frac{-\cos\theta\sin^4\theta}{5} - \frac{4\cos\theta\sin^2\theta}{15} - \frac{8\cos\theta}{15}\Big|_0\right.$$

$$= \frac{-36\sqrt{3}+64}{3} - \left(\frac{-147\sqrt{3}-256}{15}\right) = \boxed{\frac{64}{15} - \frac{11\sqrt{3}}{5}}$$

(d) $\int_0^1 \frac{x^3}{\sqrt{9-x^2}}\,dx$ let $x = 3\sin\theta \Rightarrow dx = 3\cos\theta\,d\theta$

$$= \int_{x=0}^{x=1} \frac{27\sin^3\theta}{3\cos\theta}\cdot 3\cos\theta\,d\theta$$

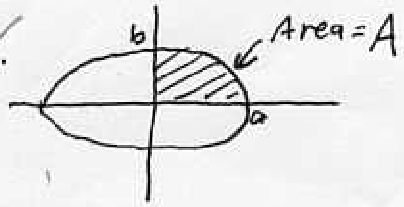
$$= 27\int_{x=0}^{x=1}\sin\theta\cdot\sin^2\theta\,d\theta = 27\int_{x=0}^{x=1}\sin\theta(1-\cos^2\theta)\,d\theta$$



let $u = \cos\theta \Rightarrow du = -\sin\theta\,d\theta$

$$= -27\int_{x=0}^{x=1}(1-u^2)\,du = -27\left[\cos\theta - \frac{\cos^3\theta}{3}\Big|_{\theta=0}^{\theta=\arcsin 1/3}\right]$$

$$= -27\left[\left(\frac{2\sqrt{2}}{3} - \frac{16\sqrt{2}}{81}\right) - \left(1 - \frac{1}{3}\right)\right] = \boxed{\frac{-38\sqrt{2}+54}{3}}$$

25. 

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y = b\sqrt{1-\frac{x^2}{a^2}}$

$\Rightarrow A = \int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx$.

let $x = a\sin\theta$

$\Rightarrow dx = a\cos\theta\, d\theta$

x	θ
0	0
a	$\pi/2$

$$\Rightarrow A = \int_0^{\pi/2} b\sqrt{1-\sin^2\theta}\cdot a\cos\theta\,d\theta$$

$$= ab\int_0^{\pi/2}\cos^2\theta\,d\theta = ab\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\Big|_0^{\pi/2}\right]$$

$$= ab\left[\left(\frac{\pi}{4}-0\right)-(0-0)\right] = \frac{ab\pi}{4}$$

$\Rightarrow$ Area of Ellipse $= 4A = \boxed{ab\pi}$ $\square$

26. (Taken from Stewart, pg 431)

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^t$$

$$= \lim_{t\to\infty}\frac{1}{1-p}\left(\frac{1}{t^{p-1}}-1\right)$$

If $p>1$, then $p-1>0 \Rightarrow t^{p-1}\to\infty$ & $\frac{1}{t^{p-1}}\to 0$

$$\Rightarrow \int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} \quad \text{for } p>1$$

But if $p\leq 1$, then $p-1<0 \Rightarrow \frac{1}{t^{p-1}} = t^{1-p}\to\infty$ and the integral diverges.

If $p=1$, then we have

$\int_1^\infty \frac{1}{x} = \lim_{t\to\infty}\ln(t) = \infty$. So the integral again diverges.

So the integral converges for $p>1$ & diverges for $p\leq 1$.