Problem Set 33

20. a) $y'' + 6y' = 7y$

$y'' + 6y' - 7y = 0$

Characteristic eqn -

$r^2 + 6r - 7 = 0$

$(r+7)(r-1) = 0$

$r_1 = -7,\ r_2 = 1$ - 2 real roots

so $y = C_1 e^{-7t} + C_2 e^t$

b) $y'' + 6y' + 9y = 0$

Characteristic eqn -

$r^2 + 6r + 9 = 0$

$(r+3)(r+3) = 0$

$r_1 = -3$ - 1 real root

so $y = C_1 e^{-3t} + C_2 t e^{-3t}$

c) $y'' + 5y' + 6y = 0$

Characteristic eqn -

$r^2 + 5r + 6 = 0$

$(r+3)(r+2) = 0$

$r_1 = -3,\ r_2 = -2$ - 2 real roots

so $y = C_1 e^{-3t} + C_2 e^{-2t}$

21. a) i. $y = C_1 e^{-7t} + C_2 e^t$

$y(0) = -2 \Rightarrow -2 = C_1 + C_2$

$y' = -7C_1 e^{-7t} + C_2 e^t$

$y'(0) = 0 \Rightarrow 0 = -7C_1 + C_2$

$7C_1 = C_2$

$-2 = C_1 + 7C_2$

$-2 = 8C_1$

$C_1 = -1/4,\ C_2 = -7/4$

so $y(t) = -\frac{1}{4}e^{-7t} - \frac{7}{4}e^t$

ii. $\lim_{t\to\infty}\left(-\frac{1}{4}e^{-7t} - \frac{7}{4}e^t\right) = -\infty$

b) i. $y = C_1 e^{-3t} + C_2 t e^{-3t}$

$y(0) = -2 \Rightarrow -2 = C_1$

$y' = -3C_1 e^{-3t} - 3C_2 t e^{-3t} + C_2 e^{-3t}$

$y'(0) = 0 \Rightarrow 0 = -3C_1 + C_2$

$\Rightarrow C_2 = -6$

so $y(t) = -2e^{-3t} - 6te^{-3t}$

ii. $\lim_{t\to\infty}\left(-2e^{-3t} - 6te^{-3t}\right) = 0$

(use L'Hospital to verify)

c) i. $y = C_1 e^{-3t} + C_2 e^{-2t}$

$y(0) = -2 \Rightarrow -2 = C_1 + C_2$

$y' = -3C_1 e^{-3t} - 2C_2 e^{-2t}$

$y'(0) = 0 \Rightarrow 0 = -3C_1 - 2C_2$

$-\frac{2}{3}C_2 = C_1$

$-2 = -\frac{2}{3}C_2 + C_2$

$\Rightarrow C_2 = -6$

$C_1 = 4$

so $y(t) = 4e^{-3t} - 6e^{-2t}$

ii. $\lim_{t\to\infty}\left(4e^{-3t} - 6e^{-2t}\right) = 0$

22. $x'' + 4x' + 3x = 0$

$x(0) = 1 \quad x'(0) = 2$

a. $r^2 + 4r + 3 = 0$

$(r+3)(r+1) = 0$

$\Rightarrow r_1 = -3, \; r_2 = -1$

$x(t) = C_1 e^{-3t} + C_2 e^{-t}$

$x(0) = 1 \Rightarrow 1 = C_1 + C_2$

$x' = -3C_1 e^{-3t} - C_2 e^{-t}$

$x'(0) = 2 \Rightarrow 2 = -3C_1 - C_2$

$C_1 = 1 - C_2$

$2 = -3(1 - C_2) - C_2$

$C_2 = 5/2, \; C_1 = -3/2$

$x(t) = -3/2\, e^{-3t} + 5/2\, e^{-t}$

b. Can x be zero?

$0 = -3/2\, e^{-3t} + 5/2\, e^{-t}$

$3/2\, e^{-3t} = 5/2\, e^{-t}$

$3/5 = e^{2t}$

since e^t is positive, $e^t = \sqrt{3/5}$

$t = \ln(\sqrt{3/5})$

$t = \frac{1}{2} \ln(3/5)$

since $\ln(3/5)$ is negative, this value for t is impossible

$\Rightarrow$ x is never zero

c. When is the mass farthest from equilibrium?

Find critical points.

$x(t) = -3/2\, e^{-3t} + 5/2\, e^{-t}$

$x'(t) = 9/2\, e^{-3t} - 5/2\, e^{-t}$

$0 = 9/2\, e^{-3t} - 5/2\, e^{-t}$

$5/2\, e^{-t} = 9/2\, e^{-3t}$

$e^{2t} = 9/5$

$e^t = \sqrt{9/5}$

$t = \frac{1}{2} \ln(9/5)$

this is a legitimite time value

find the position at this time

$x(t) = -3/2\, e^{-3(1/2 \ln(9/5))} + 5/2\, e^{-(1/2 \ln(9/5))}$

≈ 1.242

23. a. $x(t) = C_1 e^{at} + C_2 e^{bt}$

$0 = C_1 e^{at} + C_2 e^{bt}$

$-C_2 e^{bt} = C_1 e^{at}$

$-C_2/C_1 = e^{(a-b)t}$

$\ln(-C_2/C_1) = (a-b)t$

$\dfrac{\ln(-C_2/C_1)}{a-b} = t$

If C_1 and C_2 are either both negative or both positive, there is no solution. (Also, neither can be zero if there is a solution.)

If C_1 and C_2 are of opposite sign, then there is one solution.

So $x(t) = 0$ at most once.

b. $x(t) = C_1 e^{at} + C_2 t e^{at}$

Again, since e^{at} is positive and so is t, there is only a solution if C_1 and C_2 have opposite sign.

$0 = C_1 e^{at} + C_2 t e^{at}$

$$0 = (C_1 + C_2 t)e^{at}$$

$$\Rightarrow \quad C_1 + C_2 t = 0$$

$$C_2 t = -C_1$$

$$t = -C_1/C_2$$

Again, $x(t) = 0$ at most once.

c. If the characteristic equation has one real root, then we have an equation of the form $C_1 e^{at} + C_2 t e^{at}$. In part b, we showed that there is at most one time when $x = 0$.
If the characteristic equation has two real roots, then we have an equation of the form $C_1 e^{at} + C_2 e^{bt}$. In part a, we showed that there is a most one time when $x = 0$.
If the mass oscillates around the equilibrium point, then clearly it will be at position zero at many times.
Thus, these equations cannot model this oscillating spring.

24. $$e^{bti} = \sum_{n=0}^{\infty} \frac{(bti)^n}{n!} = 1 + \frac{bti}{1!} + \frac{(bti)^2}{2!} + \frac{(bti)^3}{3!} + $$

Consider terms without i
(these are the terms with even powers)

their sum is $$1 - \frac{(bt)^2}{2!} + \frac{(bt)^4}{4!} - \frac{(bt)^6}{6!} +$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (bt)^{2n}}{2n!}$$

$$= \cos(bt)$$

now consider terms with i

their sum is $$\frac{bti}{1!} - \frac{(bt)^3 i}{3!} + \frac{(bt)^5 i}{5!} + \cdots$$

$$= i\left(\sum_{n=0}^{\infty} (-1)^n \frac{(bt)^{2n+1}}{(2n+1)!}\right)$$

$$= i\sin(bt)$$

Add these to get $e^{bti} = \cos(bt) + i\sin(bt)$

$$e^{\pi i} = \cos(\pi) + i\sin(\pi)$$

$$= -1$$