Problem Set

12. In order to have a periodic solution, $x'' + bx' + cx = 0$ must have imaginary roots $\alpha \pm \beta i$. If this is true, then the solution will be of the form $y =$ $e^{\alpha t}[C_3 \cos(\beta t) + C_4 \sin(\beta t)]$, which has period $\frac{2\pi}{\beta}$.

13. $x'' + bx' + cx = 0$

Since we want a periodic solution, it must be of the form $e^{\alpha t}[C_3 \cos(\beta t) + C_4 \sin(\beta t)]$.

$x(0) = 5$

$$5 = e^0 [C_3 \cos 0 + C_4 \sin 0]$$

$$5 = C_3$$

$$x'(t) = e^{\alpha t}[-C_3\beta \sin(\beta t) + C_4 \beta \cos(\beta t)] + \alpha e^{\alpha t}[C_3 \cos(\beta t) + C_4 \sin(\beta t)]$$

$x'(0) = 0$

$$0 = C_4\beta + C_3\alpha$$

$$0 = C_4\beta + 5\alpha$$

$$C_4 = \frac{-5\alpha}{\beta}$$

if $x(n) = 5$ for all integers then the solution has period $= 1$, So β must be $\pm 2\pi$.

$$5 = e^{\alpha n}[5 \cdot \cos 2\pi n + C_4 \sin 2\pi n]$$

$$5 = e^{\alpha n} \cdot 5$$ (since cos is even, we get the same result with -2π)

$$e^{\alpha n} = 1$$

$$\alpha = 0$$ (so C_4 must be 0 as well)

$$-b = (\alpha + \beta i) + (\alpha - \beta i)$$
$$b = 0$$

$$c = (\alpha + \beta i)(\alpha - \beta i) = \alpha^2 + \beta^2 = (2\pi)^2$$

$$\boxed{x'' + (2\pi)^2 x = 0}$$

25.

a. $y'' - 9y' = 0$

$r^2 - 9r = 0$

$r(r-9) = 0$

$r_1 = 0$ and $r_2 = 9$

2 real roots, so

$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$= C_1 + C_2 e^{9t}$

b. $y'' - 9y = 0$

$r^2 - 9 = 0$

$r_1 = 3$ and $r_2 = -3$

2 real roots so

$y = C_1 e^{3t} + C_2 e^{-3t}$

c. $y'' + 9y = 0$

$r^2 + 9 = 0$

$r_1 = 3i$ and $r_2 = -3i$

imaginary roots so since $\alpha = 0$ and $\beta = 3$

$y = C_1 \cos(3t) + C_2 \sin(3t)$

d. $y'' - 9 = 0$

$y'' = 9$

$y' = 9t + C_1$

$y = \frac{9t^2}{2} + C_1 t + C_2$

e. $y'' - 2y' - y = 0$

$r^2 - 2r - 1 = 0$

$r_1 = 1 + \sqrt{2}$ and $r_2 = 1 - \sqrt{2}$

2 real roots, so

$y = C_1 e^{(1+\sqrt{2})t} + C_2 e^{(1-\sqrt{2})t}$

f. $y'' - 2y' + 2y = 0$

$r^2 - 2r + 2 = 0$

$r_1 = 1 + i$ and $r_2 = 1 - i$

imaginary roots with $\alpha = \beta = 1$

So $y = e^t [C_1 \cos t + C_2 \sin t]$

26. There are three possible situations –

1. 2 real roots – both will be negative

this is the "overdamped" case

a. $b = -(r_1 + r_2)$ $\quad c = r_1 r_2$

$\Rightarrow b > 0$ $\quad \Rightarrow c > 0$

b. $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$\lim_{t \to \infty} C_1 e^{r_1 t} + C_2 e^{r_2 t} = 0$ since r_1 and r_2 are negative

2. 1 real root - it will be negative

this is a "Critically damped" case

a. $b = -2r_1$ $\quad c = r_1^2$

$\Rightarrow b > 0$ $\quad \Rightarrow c > 0$

b. $x = C_1 e^{r_1 t} + C_2 t e^{r_1 t}$

$\lim_{t \to \infty} C_1 e^{r_1 t} + C_2 t e^{r_1 t} = 0$ because r_1 is negative

3. Complex roots $\alpha \pm \beta$, $\alpha < 0$

this is the "underdamped" case

a. $b = -2\alpha$ $\quad c = (\alpha + \beta)(\alpha - \beta)$

$\Rightarrow b > 0$ $\quad = \alpha^2 + \beta^2$

$\Rightarrow c > 0$

b. $x = e^{\alpha t}\left[C_1 \cos(\beta t) + C_2 \sin(\beta t)\right]$

$\lim_{t \to \infty} e^{\alpha t}\left[C_1 \cos(\beta t) + C_2 \sin(\beta t)\right] = 0$ because $\alpha < 0$

27.

a. $x'' + 3x' + 2x = 0$

b. $x'' - 3x' + 2x = 0$

c. $x'' + x = 0$

There are many other possible solutions. These are just examples.