MATH 1b (FALL '95)

SOLUTIONS TO THE FINAL EXAM

① a) $\int \sec^2\theta \, d\theta = \boxed{\tan\theta + C}$ (SINCE $\frac{d}{d\theta}\tan\theta = \sec^2\theta$)

b) $\int z e^{3z^2+1}\, dz = \frac{1}{6}\int e^{w}\, dw = \frac{1}{6}e^{w} + C$

$= \boxed{\frac{1}{6}e^{3z^2+1} + C}$

$w = 3z^2+1$
$dw = 6z\,dz$ so
$z\,dz = \frac{dw}{6}$

c) $\int y \sec^2(y)\, dy = y\tan y - \int \tan y\, dy = y\tan y - \int \frac{\sin y}{\cos y} dy$

$u = y \quad v' = \sec^2(y)$
$u' = 1 \quad v = \tan y$

SET $z = \cos y$
$dz = \sin y\, dy$

$= \boxed{y\tan y + \ln|\cos y| + C}$

d) $\int \frac{dt}{1+\sqrt{t}} = 2\int\left(\frac{u-1}{u}\right)du = 2\int\left(1-\frac{1}{u}\right)du$

$= 2u - 2\ln|u| + C$

$\boxed{= 2(1+\sqrt{t}) - 2\ln(1+\sqrt{t}) + C}$

$u = 1+\sqrt{t}$ so
$(u-1)^2 = t$ so
$2(u-1)\,du = dt$

e) $\int \frac{dx}{(b+ax)^2} = \frac{1}{a}\int\frac{du}{u^2} = -\frac{1}{a}u^{-1} + C$

$= \boxed{-\frac{1}{a}(b+ax)^{-1} + C}$

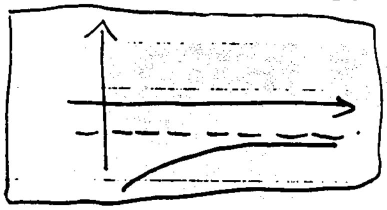
$u = b + ax$
$du = a\,dx$
$dx = \frac{du}{a}$

② THE POLLUTANTS ARE LEAVING THE LAKE AND NO NEW POLLUTANTS ARE ENTERING SO THE NET RATE <u>IS</u> THE <u>RATE OUT</u>! THE TOTAL VOLUME OUT IS 2 gals/min OF WHICH ONLY $2\cdot\frac{Q}{V}$ IS THE POLLUTANT (VOLUME × CONCENTRATION), SO

$$\boxed{\frac{dQ}{dt} = -2\frac{Q}{V} \quad \text{OR CHOICE C)}}$$

3) SINCE $f(y) \geq 0$ FOR ALL y, $\frac{dy}{dx} \geq 0$ FOR ALL y

SO $y(x)$ IS NONDECREASING FOR ALL y HENCE CHOOSE



4) $\cos t = 1 - \frac{t^2}{2!} + \ldots$ SO $\cos(3x^2) = 1 - \frac{(3x^2)^2}{2!} +$

$$= \boxed{1 - \frac{9}{2}x^4 + \ldots}$$

5) IN GENERAL, $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!}$

SO IF $a = 2$, WE HAVE

$$f(x) \approx P_3(x) = 4 + 3(x-2) - \frac{5}{2}(x-2)^2 + \frac{12}{6}(x-2)^3$$

$$= 4 + 3(x-2) - \frac{5}{2}(x-2) + 2(x-2)^3$$

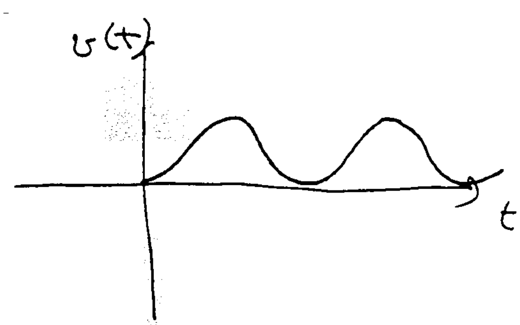
6) FROM THE PICTURE, WE SEE THAT $g(a) > 0$, $g'(a) > 0$, $g''(a) < 0$ SO

a) POSITIVE b) POSITIVE c) NEGATIVE

(as $g(x) \approx g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2$ so

$C_0 = g(a)$, $C_1 = g(a)$; $C_2 = \frac{g''(a)}{2}$)

7) a) $v(0) = 0$ AND $v(t) \geq 0$ SINCE FOR NO t IS THE AREA BETWEEN 0 AND t UNDER $a(t)$ NEGATIVE



7 (CONTINUED)

IF $s(t)$ IS THE POSITION OF THE PARTICLE, WE ARE GIVEN THAT $s(0)=0$ AND WE KNOW THAT $s(t)$ IS NONDECREASING SINCE $v(t) \geq 0$ FOR ALL t SO

8 TOTAL SPENDING (IN MILLIONS OF DOLLARS) IS

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ THIS IS A CONVERGENT GEOMETRIC SERIES WITH $a=1$, $x=\frac{1}{2}$ SO ITS SUM

$$S = \frac{a}{\ } = \frac{1}{1-\frac{1}{2}} = \boxed{2 \text{ MILLION DOLLARS}}$$

9 a) $\text{COST} = \begin{pmatrix}\text{COST PER}\\ \text{KW HOUR}\end{pmatrix} \times \begin{pmatrix}\text{NUMBER}\\ \text{OF KW}\end{pmatrix}(\text{HOURS})$ IF BOTH OF THE LATTER QUANTITIES ARE CONSTANT (WHICH THEY ARE NOT!)

SO WE PICK SOME TIME t_i AND ASSUME THAT FOR TIMES BETWEEN t_i AND $t_i + \Delta t$ THE ABOVE QUANTITIES ARE APPROXIMATELY CONSTANT ON THE SMALL TIME INTERVAL. SO LET C_i BE THE i^{th} CONTRIBUTION TO THE COST THEN

$$C_i \approx E(t_i)\,C(t_i)\cdot\Delta t = \left[5 + \sin\left(\frac{\pi t_i}{12}\right)\right]\left[0.1 + 0.01\sin\left(\frac{\pi t_i}{24}\right)\right]\cdot\Delta t$$

THEN
$$\boxed{\text{COST} \approx \sum_{i=1}^{n} C_i = \sum_{i=1}^{n} E(t_i)\,C(t_i)\,\Delta t = \sum_{i=1}^{n}\left[5 + \sin\left(\frac{\pi t_i}{12}\right)\right]\left[0.1 + 0.01\sin\left(\frac{\pi t_i}{24}\right)\right]\Delta t}$$

b) LETTING $n \to \infty$ WE GET

$$\boxed{\text{COST} = \int_0^{24}\left[5 + \sin\left(\frac{\pi t}{12}\right)\right]\left[0.1 + 0.01\sin\left(\frac{\pi t}{24}\right)\right]dt}$$

(10) a) SEPARATING VARIABLES WE GET

$$\frac{dy}{y} = \cos x \, dx \quad \text{OR} \quad \int \frac{dy}{y} = \int \cos x \, dx$$

SO $$\ln|y| = \sin x + c \quad \text{OR} \quad |y| = e^{c} \cdot e^{\sin x}$$

OR $$y = \pm e^{c} e^{\sin x} \quad \text{OR} \quad y = A e^{\sin x}$$

SINCE $y(0) = A = 4$ WE GET $\boxed{y = 4e^{\sin x}}$

b) SINCE e^{t} IS AN <u>INCREASING FUNCTION</u> IT WILL REACH ITS MINIMUM AT THE MINIMAL VALUE OF ITS INPUT AND IT WILL REACH ITS MAXIMUM AT THE MAXIMAL VALUE OF ITS INPUT SO THE MINIMUM IS REACHED WHEN $\sin x = -1$ AND

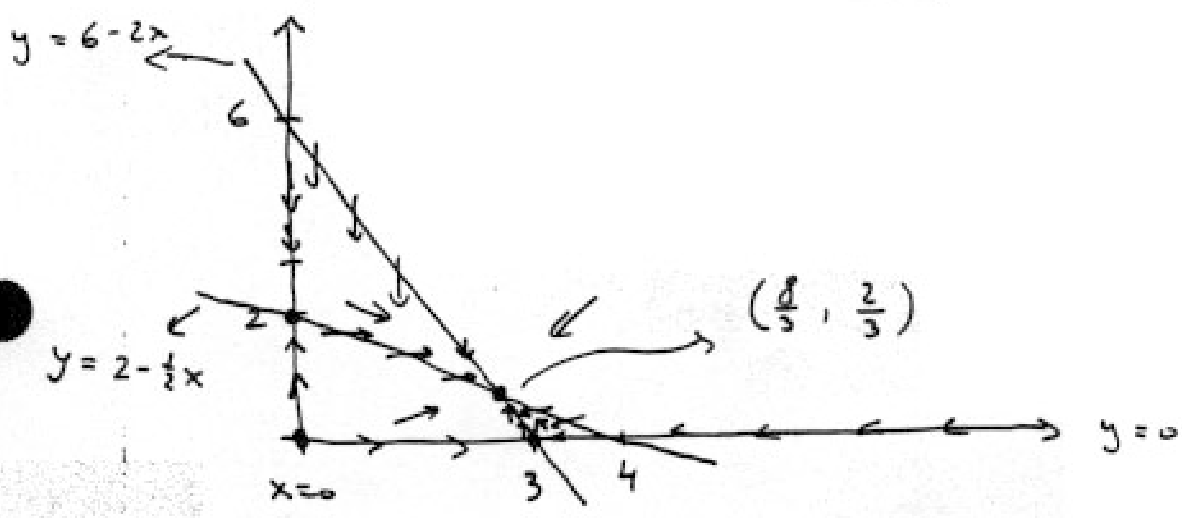
$\boxed{\text{MIN} = 4e^{-1} = \frac{4}{e}}$ AND MAXIMUM IS REACHED WHEN $\sin x =$ AND $\boxed{\text{MAX} = 4e}$

(11) a) $\frac{dx}{dt} = x(6 - 2x - y)$ SO $\frac{dx}{dt} = 0$ IF

$\boxed{x = 0 \quad \text{OR} \quad y = 6 - 2x}$

$\frac{dy}{dt} = y(4 - 2y - x)$ SO $\frac{dy}{dt} = 0$ IF

$\boxed{y = 0 \quad \text{OR} \quad y = 2 - \frac{1}{2}x}$ SO



b) THE EQUILIBRIA ARE THE POINTS WHERE BOTH $\frac{dx}{dt}$ AND $\frac{dy}{dt}$ ARE ZERO SO

$$(0,0) \;,\; (0,2) \;,\; (3,0) \;,\; \left(\tfrac{8}{3}, \tfrac{2}{3}\right)$$

c) i) THEY COMPETE

d) IN THE LONG RUN THE TWO POPULATIONS TEND TO $\left(\frac{8}{3}, \frac{2}{3}\right)$, REGARDLESS OF INITIAL CONDITIONS.

12 a) IF $C = 6$ THE CHARACTERISTIC EQUATION IS

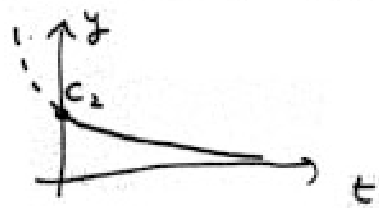
$$2\lambda^2 + 6\lambda + 4 = 0 \quad \text{OR} \quad 2(\lambda+2)(\lambda+1) = 0$$

SO $\lambda = -2$ OR $\lambda = -1$; HENCE

$$y(t) = C\, e^{-2t} + C_2\, e^{-t}$$

A POSSIBLE SOLUTION (IF $C = 0$) IS $C_2 e^{-t}$

OR



b) IF $C = 5$ THEN THE CHARACTERISTIC EQUATION IS

$$2\lambda^2 + 5\lambda + 4 = 0 \quad \text{OR}$$

$$\lambda_{1,2} = \frac{-5 \pm \sqrt{25 - 32}}{4} = \frac{-5 \pm \sqrt{-7}}{4} = -\frac{5}{4} \pm \frac{\sqrt{7}}{4} i$$

SO $$y(t) = C\, e^{-\frac{5}{4}t} \cos\left(\frac{\sqrt{7}}{4} t\right) + C_2\, e^{-\frac{5}{4}t} \sin\left(\frac{\sqrt{7}}{4} t\right)$$

12 CONTINUED

c) FOR $0 \le c \le \sqrt{32}$ SINCE WE WANT $\sqrt{c^2 - 32}$ TO BE COMPLEX i.e.

$c^2 - 32 < 0$, THIS IS TRUE FOR

$-\sqrt{32} < c < \sqrt{32}$, BUT SINCE WE ARE

GIVEN THAT $c \geq 0$, WE CONCLUDE

$$\boxed{0 \le c < \sqrt{32}}$$

d) SUPPOSE c IS BETWEEN 0 AND $\sqrt{32}$ THEN THE CHARACTERISTIC EQUATION

$2r^2 + cr + 4 = 0$ HAS SOLUTIONS

$$\frac{-c \pm \sqrt{c^2 - 32}}{4} \quad \text{OR} \quad \frac{-c \pm \sqrt{32 - c^2}\, i}{4}$$

SO THE SOLUTIONS TO THE DIFF. EQN ARE OF THE FORM

$$y(t) = C_1 e^{-\frac{c}{4}} \cos\left(\frac{\sqrt{32-c^2}}{4} t\right) + C_2 e^{-\frac{c}{4}} \sin\left(\frac{\sqrt{32-c^2}}{4} t\right)$$

i) THE PERIOD $= \dfrac{2\pi}{} = \dfrac{8\pi}{\sqrt{32 - c^2}}$

ii) PERIOD IS MINIMIZED BY THE LARGEST POSSIBLE DENOMINATOR i.e. BY $c = 0$ SO $\dfrac{8\pi}{\sqrt{32}}$ IS THE MINIMAL PERIOD

iii) WANT $\dfrac{8\pi}{\sqrt{32 - c^2}} = 0$ SO $\dfrac{64\pi^2}{100} = 32 - c^2$ SO $\boxed{c \approx 5.07}$