Abbreviated Answers to the Fall '99 1b Final (1/21/00)

These are not full sol'ns - just answers plus...

1. a) converges - by ratio test

b) converges - compare with $\sum \frac{n}{n^3} = \sum \frac{1}{n^2}$

c) converges by Alt. Series Test

2. Use ratio test to get: converges for $|x-1|<1$ i.e. $0<x<2$.

Check endpts: At $x=0$ $\sum \frac{(-1)^n}{\sqrt{n+n^2}}$ converges by Altern. Series test

At $x=2$ $\sum \frac{1}{\sqrt{n+n^2}}$ diverges (it's like $\frac{1}{n}$: you can't use direct comparison - must use Limit comparison - which we didn't cover!)

3a. Use parts: $u=\ln x$ $dv = x^{1/2}dx$

Get $\frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2}+C$

b) u-sub. let $u=e^{\sqrt{x}}+1$ $\Rightarrow \int \frac{2\,du}{u} = 2\ln|u|+C = 2\ln|e^{\sqrt{x}}+1|+C$

$du = e^{\sqrt{x}}\cdot\frac{1}{2}x^{-1/2}dx$

4. Improper at $x=-1$, $x=2$ so compute

$$\int_{-7}^{-1}\frac{dx}{(x-2)(x+1)} + \int_{-1}^{0}\frac{dx}{(x-2)(x+1)} + \int_0^2 — + \int_2^8 —$$

If any integral diverges the whole thing diverges.

Use partial fractions: $\frac{1}{(x-2)(x+1)} = \frac{\frac{1}{3}}{x-2} - \frac{\frac{1}{3}}{x+1}$ so $\int\frac{dx}{(x-2)(x+1)} = \frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C$

$= \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + C$

Compute $\lim_{b\to -1^-}\int_{-7}^{b}\frac{dx}{(x-2)(x+1)} = \lim_{b\to-1^-}\frac{1}{3}\ln\left|\frac{b-2}{b+1}\right| - \frac{1}{3}\ln\frac{9}{6}$

$= \lim_{b\to-1^-}\underbrace{\frac{1}{3}\ln\left(\text{this} \to 0\right)}_{\to -\infty} - \frac{1}{3}\ln\frac{3}{2} \Rightarrow$ diverges

so the whole integral diverges.

5. $\pi\int_0^1(\text{radius})^2\,dy = \pi\int_0^1(e^y)^2\,dy = \cdots \quad \frac{\pi(e^2-1)}{2}$

6. $-\tan x$ is the derivative of $f(x)$ - so must integrate term by term to find $f(x)$

$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$

$f'(x) = -x - \frac{x^3}{3} - \frac{2x^5}{15} - \cdots$

$f(x) = \left(-\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} + \cdots\right) + C$

$f(0) = \ln(\cos 0) = \ln 1 = 0 \Rightarrow C=0$ $\Rightarrow$ 1st 3 non-zero terms: $-\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45}$

7. $y = \sqrt{(1-x^{2/3})^3} = (1-x^{2/3})^{3/2}$

$y' = \ldots = -\frac{\sqrt{1-x^{2/3}}}{x^{1/3}}$ so $(y')^2 = \frac{1-x^{2/3}}{x^{2/3}} = x^{-2/3}-1$ and $(y')^2+1 = x^{-2/3}$

Arclength $= \int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1\sqrt{x^{-2/3}}\,dx = \int_0^1 x^{-1/3}\,dx = \ldots = \frac{3}{2}$

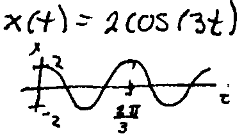
8. 1st order linear - need an integrating factor: we didn't cover this: Ans: $y = -1+4e^{-\frac{1}{2}x}$

9. a) $\frac{dy}{dt} = 5 - \frac{y}{2}$ b) $y(0)=0$ c) Separate variables $\Rightarrow \ldots y = 10 - Ce^{-t/2}$

But $y(0)=0 \Rightarrow y(t) = 10-10e^{-t/2}$ d) $y(2\ln 2) = \ldots = 5 \Rightarrow 5$ ounces.

10 a) $x(t) = C_1\cos 3t + C_2\sin 3t$. Use initial conditions to get $x(t) = 2\cos(3t)$.

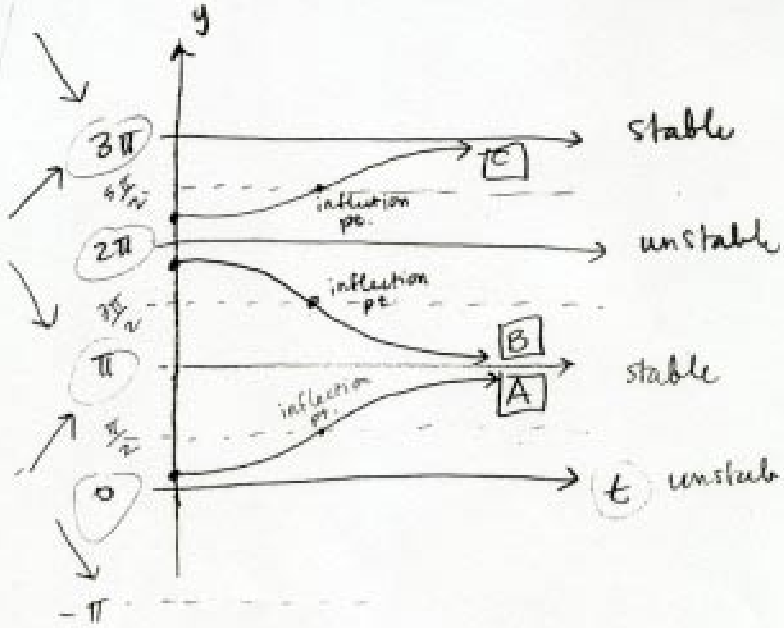
b) Spring has no friction: oscillates between 2 & -2 forever.



c) To be critically damped means 1 real root $\Rightarrow b^2-4c=0$.

$b^2-4(9)=0 \Rightarrow b=6$: $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$

14 (a) and (b)



Solution curves

$\boxed{A}$ $y(0) = 0.1$

$\boxed{B}$ $y(0) = 6$

$\boxed{C}$ $y(0) = 7$

(c)

π is a stable equilibrium solution and the curve from $y(0) = 0.1$ approaches π as $t \to \infty$

12.

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + 7a_7$$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + 42a_7$$

$$x \cdot y = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \ldots$$

$$y'' - xy = (2a_2) + (6a_3 - a_0)x + (12a_4 - a_1)x^2 + (20a_5 - a_2)x^3 + (30a_6 - a_3)x^4 = 0$$

$y(0) = a_0 = 1$

$y'(0) = a_1 = 0$

$a_2 = 0$

$a_3 = \frac{1}{6}$

$a_4 = 0$

$a_5 = 0$

$a_6 = \frac{1}{180}$

First 3 nonzero terms

$$y(x) = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6$$