Solutions to the Second Exam for Mathematics 1b

April 15, 2004

1. (a) answers: (ii) and (iii).

   (iii) is the definition of convergence. If a series converges, then its terms must approach 0.

   Neither (v) nor (vi) need be true. For example, think about an alternating series.

   (i) is definitely false.

   (b) i. Not necessarily true. For example, consider $\sum_{k=1}^{\infty} \frac{1}{k}$. This series, the harmonic series, diverges (this can be proven using the integral test), and yet $\lim_{k\to\infty} \frac{1}{k} = 0$.

   ii. Not necessarily. For example, the Taylor series for $f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ converges only for $|x| < 1$.

   (Note: some people thought that the series $\sum_{k=1}^{\infty} a_k x^k$ is necessarily a geometric series. This is not so. Let $f(x) = e^k$ for instance. $a_k = \frac{1}{k!}$ and the series is not geometric (and converges for all x.)

   (c) The center of the series is -3. The radius of convergence is greater than or equal to 4 and less than or equal to 5. Recall that at the endpoints $-3-R$ and $-3+R$ the series may converge or may diverge.

   Therefore, the series is certain to converge for $x \in (-7, 1]$.

   The series is certain to diverge for $x \geq 2$ or $x < -8$.

2. (a) Converges, by direct comparison to $\sum_{n=1}^{\infty} \frac{3^n}{5^n}$.

   $\frac{3^n}{5^n+3n} < \frac{3^n}{5^n}$. $\sum_{n=1}^{\infty} \frac{3^n}{5^n}$ converges, since it is a geometric series with $|r| = 3/5 < 1$.

   (b) Diverges, by limit comparison to $\sum_{n=1}^{\infty} \frac{1}{n}$.

   $\lim_{n\to\infty} \frac{\frac{n^3}{3n^4+n^2+5}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3}{3n^4+n^2+5} \cdot \frac{n}{1} == \lim_{n\to\infty} \frac{n^4}{3n^4+n^2+5} = \frac{1}{3}$. This limit is finite and non-zero, and the harmonic series diverges, so our original series diverges.

   (c) Diverges, by n-th term test. $\lim_{n\to\infty} \frac{n}{5\ln n} = \infty$, so $\lim_{n\to\infty} (-1)^n \frac{n}{5\ln n} \neq 0$. The series diverges. How do we know $\lim_{n\to\infty} \frac{n}{5\ln n} = \infty$? We can look at $\lim_{x\to\infty} \frac{x}{5\ln x}$ and, applying L'Hopital's Rule, get $\lim_{x\to\infty} \frac{1}{5/x} = \lim_{x\to\infty} x/5 = \infty$.

   (d) Converges, by the alternating series test. The series is alternating, the terms are decreasing in magnitude, and $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} = 0$.

   (e) Consider $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^{3/2}}$ and show that this series converges.

   $\frac{|\sin n|}{n^{3/2}} < \frac{1}{n^{3/2}}$.

   $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges: ;it is a p-series with $p = 3/2 > 1$. Therefore, by direct comparison, $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^{3/2}}$ converges. We conclude that $\sum_{n=1}^{\infty} \frac{\sin n}{n^{3/2}}$ converges absolutely, and hence converges.

3. (a) We know

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \ldots + u^n + \ldots$$

for $|u| < 1$. Replace u by $-x^4$, we get

$$\frac{1}{1+x^4} = 1 - x^4 + x^8 - x^{12} + \ldots + (-1)^n x^{4n} + \ldots$$

Therefore,

$$\frac{x}{1+x^4} = x - x^5 + x^9 - x^{13} + \ldots + (-1)^n x^{4n+1} + \ldots$$

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(b) We know that $1 + u + u^2 + u^3 + \ldots + u^n + \ldots$ converges if $|u| < 1$. So the above series converges if $|-x^4| < 1$, which implies that the radius of convergence $R = 1$.

(c) Use the Taylor series in (a) and integrate term by term, we get

$$\int_0^{0.1} \frac{x}{1+x^4} dx$$

$$= \int_0^{0.1} x dx - \int_0^{0.1} x^5 dx + \int_0^{0.1} x^9 dx + \ldots + \int_0^{0.1} (-1)^n x^{4n+1} dx + \ldots$$

$$= \frac{0.1^2}{2} - \frac{0.1^6}{6} + \frac{0.1^{10}}{10} + \ldots + (-1)^n \frac{0.1^{4n+2}}{4n+2} + \ldots$$

(d) The series in (c) is alternating, its terms are decreasing in magnitude, and its terms are going towards zero. The error is less than the first unused term. We see that $\frac{0.1^{10}}{10} = 10^{-11} < 10^{-10}$. So

$$\frac{0.1^2}{2} - \frac{0.1^6}{6}$$

gives an estimate of the definite integral with error of less than 10^{-10}. And since the next term is positive, the estimate is a little smaller than the actual value.

4. (a) We apply the ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-4|^{n+1}}{2^{n+1}\sqrt{n+1}} \frac{2^n\sqrt{n}}{|x-4|^n} = \frac{|x-4|}{2} \frac{\sqrt{n}}{\sqrt{n+1}}.$$

The limit of the last expression as $n \to \infty$ is $\frac{|x-4|}{2}$, hence the power series converges for

$$|x-4| < 2,$$

or equivalently, for $2 < x < 6$. It diverges for $|x-4| > 2$. To establish the behavior at the endpoints, note that for $x = 2$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}},$$

which converges by the alternating series test, and for $x = 6$ the series is

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

which is a divergent p-series.

(b) We again take the ratio

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x+1|^{n+1}3^{n+1}}{(2n+2)!}\frac{2n!}{|x+1|^n 3^n} = \frac{3|x+1|}{(2n+1)(2n+2)}.$$

The limit of the last expression as $n \to \infty$ is 0, which is less than 1, hence the series converges for all x.

5. (a) You have two choices here. You can take derivatives, evaluate at $x = 1$, and obtain the 3rd degree Taylor series "from scratch" or you can use the binomial series with a little cleverness. In either case, you should get

$$T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$$

Below we show how to use the binomial series to get this result. Use the binomial series for $(1+u)^k$ where $u = x+1$ and $k = 1/2$ as shown below.

$$\sqrt{x} = \sqrt{1+(x-1)} = \sum_{n=0}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!}(x-1)^n.$$

So the 3rd order Taylor polynomial at $x = 1$ is

$$T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$$

(b) Plug in $x = 1.1$ into $T_3(x)$, we get

$$1 + \frac{1}{2}(0.1) - \frac{1}{8}(0.1)^2 + \frac{1}{16}(0.1)^3$$

is an good estimate of $\sqrt{0.1}$.

(c) The infinite Taylor series in (a) when $x = 1.1$ is alternating(after the second term) - and the terms are decreasing in magnitude and going towards zero. So the error is less than the first unused term. Therefore a good upper bond of the error is the next term which is

$$\left|\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}(0.1)^4\right| = \frac{5}{2^7}(0.1)^4.$$

(d) Since $\sqrt{x}$ is not differentiable at $x = 0$. The Taylor series at $x = 0$ does not exist. (In particular, the tangent line is vertical at $x = 0$, and therefore is useless in giving us an approximation.)

6. The convergence of $\sum_{n=1}^{\infty} a_n$ implies that $\lim_{n\to\infty} a_n = 0$.

(a) The series diverges by the nth term test, since

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} a_n} = \infty.$$

(b) The series converges. Indeed for all large n, we'll have

$$a_n < 1,$$

because $\lim_{n\to\infty} a_n = 0$. Thus we will have

$$a_n^3 < a_n$$

for all large n, hence by comparison $\sum_{n=1}^{\infty} {a_n}^3$ converges.

(c) The series diverges by the nth term test: since $\lim_{n\to\infty} a_n = 0$, it follows that $\lim_{n\to\infty} e^{a_n} = 1$, so the nth term does not go to zero.

(d) The series converges absolutely, therefore converges.

(e) The series could converge or could diverge. Note that for $a_n < 1$, $\sqrt{a_n} > a_n$. So, for instance, if $a_n = \frac{1}{n^2}$, then

$$\sum_{n=1}^{\infty} \sqrt{a_n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges. On the other hand, if $a_n = \frac{1}{n^4}$, then

$$\sum_{n=1}^{\infty} \sqrt{a_n} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges.