Solutions to the First Exam for Mathematics 1b

March 4, 2003

1. (a) Change of variables $u = x^2$:

$$\int_0^\infty \frac{x}{1+x^2} dx = \lim_{t\to\infty} \int_0^t \frac{x}{1+x^2} dx = \lim_{t\to\infty} \frac{1}{2} \int_0^{t^2} \frac{1}{1+u} du = \lim_{t\to\infty} \frac{1}{2} \ln(1+t^2)$$

The last limit equals infinity, so the integral diverges:

(b) Change of variables $u = \sin x$ (notice $\sin(\pm\pi/2) = \pm 1$) :

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x}{\sin x} dx = \int_{-1}^{1} \frac{1}{u} du.$$

This is the integral of an odd function, but we cannot conclude the integral is zero because it is an improper integral. We have to break it down into to pieces:

$$\int_{-1}^{1} \frac{1}{u} du = \int_{-1}^{0} \frac{1}{u} du + \int_{0}^{1} \frac{1}{u} du.$$

The first integral diverges:

$$\int_{-1}^{0} \frac{1}{u} du = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{u} du = \lim_{t\to 0^-} \ln|t| - \ln|-1| = -\infty,$$

so the original integral diverges as well.

(c) The volume of rotation equals (using vertical slices):

$$V = \int_e^\infty \pi \frac{1}{x(\ln x)^2} dx$$

After a change of variables $u = \ln x$, the integral becomes:

$$V = \pi \int_1^\infty \frac{1}{u^2} du = \pi \lim_{t\to\infty} (1 - 1/t) = \pi,$$

so the integral converges to π.

2. Density varies with the distance from the center of the pizza, so partition the interval $[0, 15]$ into n equal pieces, each of length Δx with $x_k = k\Delta x$ for $k = 0, 1, 2, \ldots n$. This partitions the pizza into co-centric rings.

The amount of capsaicin on the ith ring $\approx$ (density in mg/ cm^2) (area of ring in cm^2)

The area of the ring $\approx 2\pi r_i \Delta x$ and $r_i = x_i$.

So amount$_i \approx 3\, e^{-x_1^2/100} \cdot 2\pi x_i \Delta x \,\text{cm}^2$

$$\text{Total amount of capsaicin} = \lim_{n\to\infty} \sum_{i=1}^{n} 3\, e^{-x_i^2/100} \cdot 2\pi x_i \Delta x = \int_0^{15} 3\, e^{-x^2/100} \cdot 2\pi x\, dx$$

$$\int_0^{15} 3\, e^{-x^2/100} \cdot 2\pi x\, dx = 6\pi \int_0^{15} e^{-x^2/100} \cdot x\, dx$$

The integral can be done by simple substitution: let $u = x^2/100$. Then $du = \frac{x}{50}$ so $x\,dx = 50/, du$.

When $x = 0$, $u = 0$ and when $x = 15$, $u = (15)^2/100 = 2.25$

$$6\pi \int_0^{15} e^{-x^2/100} \cdot x\, dx = 300\pi \int_0^{2.25} e^{-u}\, du = 300\pi(-e^{-u})|_0^{2.25} = 300\pi(-e^{-2.25} + 1)$$

Answer : $300\pi(1 - e^{-2.25})$ mg of capsacian per pizza.

3. Let y be the distance measured from the ground. Slice the anthill into discs, according to the height y. If y_i is the height of the ith slice in meters, then the ith slice is a disc of radius $\sqrt{1-y_i^2}$ meters. The volume of the ith slice is

$$\text{Volume of } i\text{th slice} = \pi\left(\sqrt{1-y_i^2}\right)^2 \Delta y\, m^3 = \pi(1-y_i^2)\Delta y\,\text{m}^3$$

The weight of the ith slice (or the force needed to lift it) is:

$$\pi(1-y_i^2)\times 50\times g\text{ Newtons} = 500\pi(1-y_i^2)\Delta y\,\text{Newtons.}$$

Work done to lift the ith slice is

FORCE × DISTANCE LIFTED,

which is

$$500\pi(1-y_i^2)\Delta y\times y_i\text{ Joules.}$$

Total work done is the integral

$$500\pi\int_0^1(1-y^2)y\,dy\text{ Joules.}$$

Write the integrand as $y-y^3$, or use substitution, to arrive at the answer $125\pi\,J$.

4 (a) Slicing x from $x=0$ to $x=2$ into n equal pieces will give us cylindrical shells, or tubes. The volume of an individual slice will be approximately

$$2\pi r\,h\,\Delta x = 2\pi x_i(x_i^2+1)\Delta x$$

since the radius of the i^{th} shell is $r=x_i$ and the height is the value of the function $h=y=x_i^2+1$. Then the Riemann sum is

$$V\approx\sum_{k=1}^{n}2\pi x_i(x_i^2+1)\Delta x$$

and the integral is

$$V = 2\pi\int_0^2 x(x^2+1)dx = 12\pi$$

If we choose to slice y instead, we need to do two integrals. From $y=0$ to $y=1$ our slices will be disks (i^{th} disk's volume $=\pi 2^2$), and from $y=1$ to $y=5$ our slices will be washers with the i^{th} washer's volume $=\pi[2^2-(\sqrt{y_i-1})^2]$. Then the integral would be

$$V=\int_0^1 4\pi\,dy+\int_1^5\pi\left(2^2-\left(\sqrt{y-1}\right)^2\right)\,dy$$

(b) The jello mold must be sliced horizontally since the density of pears, $g(y)$ varies with height y from the refrigerator shelf. So we slice y from $y=0$ to $y=3$. The slices are washer shapes like pineapple slices, and the volume of the i^{th} slice will be

$$V_i\approx\pi(R^2-r^2)\Delta y = \pi\left(\left(5+\sqrt{\frac{4}{3}y_i}\right)^2-\left(5-\sqrt{\frac{4}{3}y_i}\right)^2\right)\Delta y$$

So the amount of pears in the i^{th} slice will be approximately

$$V_i\cdot g(y_i)\approx\pi\left(\left(5+\sqrt{\frac{4}{3}y_i}\right)^2-\left(5-\sqrt{\frac{4}{3}y_i}\right)^2\right)g(y_i)\Delta y$$

and the total amount will be

$$\text{Amt.} = \int_0^3\pi\left(\left(5+\sqrt{\frac{4}{3}y}\right)^2-\left(5-\sqrt{\frac{4}{3}y}\right)^2\right)g(y)\,dy = \frac{40\sqrt{3}}{3}\pi\int_0^3\sqrt{y}\,g(y)\,dy$$

5. Let's measure the distance x the bucket has been raised from the ground upwards. The bucket gains 5 pounds over 200 feet, so it gains $\frac{1}{40}$ pound per foot raised. Adding this to the initial weight of the bucket, we find

$$F_{\text{bucket}} = 5 + \frac{x}{40}$$

The weight of the rope is $\frac{1}{10}$ pound per foot, times $200 - x$ feet (the length of the rope), for a total of

$$F_{\text{rope}} = \frac{1}{10}(200 - x) = 20 - \frac{x}{10}$$

so that the total weight being lifted after the bucket is lifted x feet is

$$F_{\text{tot}} = F_{\text{bucket}} + F_{\text{rope}} = 25 - \frac{3x}{40}.$$

If we do a Riemann sum to approximate the total work, this weight would be lifted through a distance of Δx between two intervals, so we find that the work in the i'th interval is

$$W_i = F_{\text{tot}} * \Delta x.$$

so that, when we take the limit as the width of the intervals goes to 0, we get an integral:

$$\begin{aligned} W_{\text{tot}} &= \int_0^{200} F_{\text{tot}}\,dx \\ &= \int_0^{200} (25 - \frac{3x}{40})\,dx \\ &= \left(25x - \frac{3x}{80}\right)\Big|_0^{200} \\ &= 5000 - 1500 = 3500\,\text{ft-lb}. \end{aligned}$$

6. (a) $\int_1^\infty f(x)\,dx$ could converge or could diverge.
   Let $f(x) = 1/x$.
   To compute $\int_1^\infty \frac{1}{x}\,dx$ look at $= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln b - \ln 1 = \lim_{b\to\infty} \ln b = \infty$
   Therefore $\int_1^\infty \frac{1}{x}\,dx$ diverges.
   On the other hand, let $f(x) = \frac{1}{x^2}$.
   To compute $\int_1^\infty \frac{1}{x^2}\,dx$ look at $= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} -\frac{1}{x}|_1^b = \lim_{b\to\infty} -\frac{1}{b} + 1 = 1$.
   Therefore $\int_1^\infty \frac{1}{x^2}\,dx$ converges.

   (b) $\lim_{x\to\infty} f(x) = .000001 \neq 0$. Therefore the improper integral must diverge. (To reason this out, think about the area of a rectangle with height .000001 and a base of infinite length. Alternatively, compare $f(x)$ to $\frac{1}{x}$. For x large enough, $f(x) > \frac{1}{x}$, since $\lim_{x\to\infty} f(x) = .000001$ while $\lim_{x\to\infty} \frac{1}{x} = 0$.

   (c) We know that $0 < f(x) < 1$ so $[f(x)]^3 < f(x)$.
   $\int_1^\infty f(x)\,dx$ converges. Since $[f(x)]^3$ is positive, we know that as b increases $\int_1^b [f(x)]^3\,dx$ increases. But we know that for $b > 1$, $\int_1^b [f(x)]^3\,dx < \int_1^b f(x)\,dx$ and $\int_1^\infty f(x)\,dx$ converges. Therefore the integral in question converges by comparison.