Series Handout A

1. Determine which of the following sums are geometric. If the sum is geometric, express the sum in closed form.

   a) $\sum_{k=1}^{70}(\frac{1}{k})$ b) $\sum_{k=1}^{50}(\frac{1}{k})^2$ c) $\sum_{k=1}^{60}(\frac{1}{k})^k$ d) $\sum_{k=1}^{60}(1.01)^{k/3}$

2. Find the sum of each of the following.

   a) $\sum_{k=0}^{100}(\frac{1}{3})^k$ b) $\sum_{k=0}^{\infty}(\frac{1}{3})^k$ c) $\sum_{k=2}^{100}(\frac{1}{3})^k$ d) $\sum_{k=2}^{\infty}(\frac{1}{3})^k$

3. Do the first few terms of a series affect whether or not the series converges?

   Do the first few terms of a convergent series affect its sum?

4. (a) For what values of x does the series $\sum_{k=0}^{\infty} x^k$ converge? For these values of x, what function does the series converge to?

   (b) For what values of x does the series $\sum_{k=0}^{\infty}(x-4)^k$ converge? For these values of x, what function does the series converge to?

5. *What do you think?*

   Some of our problem sets will include *What do you think?* questions. You will receive full credit on your homework for a thoughtful answer to such a question - regardless of whether you have answered it correctly. Write your thoughts in pen and leave some blank space below your answer on your homework. These questions will be discussed in class and you can correct any mistaken ideas you had in pencil below.

   The next two problems ask you to make sense of the definition of a convergent series.

   Let $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ be an infinite series and let s_n denote its nth partial sum:

   $s_n = a_1 + a_2 + a_3 + \cdots + a_n$.

   Definition:
   We say $\sum_{k=1}^{\infty} a_n$ **converges** if $\lim_{n\to\infty} s_n = S$ for a finite number S.
   We write $\sum_{k=1}^{\infty} a_n = S$ and say $\sum_{k=1}^{\infty} a_n$ *converges to* S.

   If $\lim_{n\to\infty} s_n$ does not exist (or is not finite) then we say the series $\sum_{k=1}^{\infty} a_n$ **diverges**.

   Suppose you know that the infinite series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ converges and that $a_k > 0$ for k any positive integer. For each of the following statements, determine whether the statement must be true, could possibly be true, or must be false.

   (a) $\lim_{n\to\infty} a_n = 0$

   (b) $\lim_{n\to\infty} s_n = 0$

   (c) There exists a number M such that $s_n < M$ for all n.
   (This is equivalent to saying that the partial sums are bounded.)

   (d) $\sum_{k=5}^{\infty} a_k$ converges

6. *What do you think?*

   Suppose you know that $\lim_{n\to\infty} b_n = 0$. Does it necessarily follow that the infinite series $\sum_{k=1}^{\infty} b_k$ converges?

7. Suppose $a_n = f(n)$ where f is a function defined on $(-\infty, \infty)$ and n is an integer. We know that if $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$ (so $\lim_{n\to\infty} a_n = L$) as well.

   Zenobia has the mistaken belief that if $\lim_{n\to\infty} f(n)$ and $\lim_{x\to\infty} f(x)$ are always equal. Which of the following functions would be best to use to persuade her to change her mind?

   (a) $f(x) = \sin x$ (b) $f(x) = \frac{\sin x}{x}$ (c) $f(x) = \sin(\pi x)$

8. *What do you think?* Write out the first few terms of the series

$$\sum_{k=1}^{\infty} \frac{1}{2^k + k}.$$

(This series is not a geometric series.) Now write out the first few terms of the geometric series $\sum_{k=1}^{\infty} \frac{1}{2^k}$. By comparing the terms of the two series, determine whether or not the former series converges. Explain your reasoning in words carefully and clearly. Your answer will form the launching pad for the next class.

9. In class we studied a family of series known as $p-$series, series of the form $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$.
We know that $\int_1^{\infty} \frac{1}{x^p}\, dx$ diverges for $p < 1$ and converges for $p > 1$. We use the integral test to conclude that $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$ converges if $p > 1$ and diverges if $p < 1$.
Sometimes students *incorrectly* think that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is ' the dividing line' for series of any sort, not just for $p-$series. This problem is meant to dispel that notion.
   (a) We know that $\frac{1}{n+2} < \frac{1}{n}$ for all positive n. Nevertheless, show that the series $\sum_{n=1}^{\infty} \frac{1}{n+2}$ diverges.
   (b) We know that $\frac{1}{\sqrt{n^2+10}} < \frac{1}{n}$ for all positive n. Nevertheless, show that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+10}}$ diverges.
   (c) We know that $\sum_{n=2}^{\infty} \frac{1}{n\ lnn} < \frac{1}{n}$ for all positive n. Show, nevertheless, that the series $\sum_{n=2}^{\infty} \frac{1}{n\ lnn}$ diverges.
   (d) (Extra credit) In fact, $\sum_{n=2}^{\infty} \frac{1}{n\ lnn}$ diverges, but $\sum_{n=2}^{\infty} \frac{1}{n(\ lnn)^p}$ converges for any $p > 1$. Show this.

10. If f is a function, then $P_n(x)$, the Taylor polynomial about $x = 0$ associated to f, is to be thought of as a "good" approximation of f for x near zero. Let's consider the polynomial

$$f(x) = x^3 - 4x^2 + 4x.$$

   (a) What do you think the best linear approximation of this polynomial (approximation of the form $P_1(x) = a_0 + a_1 x$) ought to be at the point $b = 0$? Why?
   What is the best quadratic approximation of the polynomial f (approximation of the form $P_2(x) = a_0 + a_1 x + a_2 x^2$) at the point $b = 0$?
   What is the best cubic approximation of the polynomial f (approximation of the form $P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$) at the point $b = 0$? Does this surprise you? Why or why not?
   (b) The number 2 is a critical point of f. If we write $P_n(x) = a_0 + a_1(x-2) + a_2(x-2)^2 + ...a_n(x-2)^n$ then a_1 will be zero. Why?
   In addition, a_4, a_5, ..., a_n will all be zero. Why is that?
   (c) If b is any real number and we write $P_n(x) = a_0 + a_1(x-b) + a_2(x-b)^2 + ... + a_n(x-b)^n$ then what can we say about a_4, a_5 .....? Why?

11. Let $f(x)$ be a function defined on the domain $(-\infty, \infty)$ with exactly one zero at $x = -3$. The first and second derivatives of f exist everywhere on its domain. We know that f is increasing on $(-\infty, 1]$, decreasing on $[1, \infty)$, concave down on $(-\infty, 2)$, and concave up on $(2, \infty)$.
Below are second degree Taylor polynomials for f centered at 0, 1, 2, 3, and -4 respectively. Use the information to determine the sign (positive, negative, or zero) or each of the coefficients.
   (a) $a_0 + a_1 x + a_2 x^2$
   (b) $b_0 + b_1(x-1) + b_2(x-1)^2$
   (c) $c_0 + c_1(x-2) + c_2(x-2)^2$
   (d) $d_0 + d_1(x-3) + d_2(x-3)^2$
   (e) $e_0 + e_1(x+4) + e_2(x+4)^2$

12. By taking derivatives, show that the Taylor series for the function $f(x) = \frac{1}{1-x}$ is the geometric series $\sum_{k=0}^{\infty} x^k$. (This illustrates the fact that if a function has a power series expansion then that power series expansion *is* its Taylor series.)

13. We know that the Taylor series generated by e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. In fact, it can be shown that this series converges to e^x for all x and gives an alternate representation of e^x. In other words, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. By evaluating both sides of this equation at $x = -0.1$ you can obtain an expression for $e^{-0.1}$. How many terms of the series are needed to approximate $e^{-0.1}$ with error less than 10^{-8}?

14. If $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ for all x, then how many non-zero terms of the series expansion for $\sin(0.2)$ must be used in order to approximate $\sin(0.2)$ with error less than 10^{-6}? Will this approximation be too large, or too small?

15. The Alternating Series Test says that if an infinite series is (i) alternating, (ii) the magnitude of the terms is decreasing, and (iii) the magnitude of the terms tends to zero, then the series converges.

    This is a test for convergence only; **don't** try use it to show divergence. If the terms are not tending to zero, then the series diverges by the Nth term test for Divergence, not by the AST. If either one of the other two conditions is not satisfied then the test is *inconclusive* - meaning the series might converge or it might diverge.

    Below (for your reading pleasure) we present an alternating series whose terms are tending towards zero but not decreasing in magnitude. Conditions (i) and (iii) are satisfied by the series

$$1 - \frac{1}{2} + \frac{2}{2} - \frac{1}{3} + \frac{2}{3} - \frac{1}{4} + \frac{2}{4} - \cdots + \frac{2}{n} - \frac{1}{n} + \cdots$$

    but

$$1 + \left(-\frac{1}{2} + \frac{2}{2}\right) + \left(-\frac{1}{3} + \frac{2}{3}\right) + \left(-\frac{1}{4} + \frac{2}{4}\right) + \cdots + \left(\frac{2}{n} - \frac{1}{n}\right) + \cdots$$

    can be written

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n} + \cdots$$

    which is the harmonic series. The harmonic series diverges, so the series displayed diverges.

    In the problems below, determine whether or not the series converges. (The alternating series test cannot be invoked: why not?)

    (a) $\sum_{k=1}^{\infty} (-1)^k \frac{2k^2 - 10k}{10k^2 + 5k}$

    (b) $\sum_{k=1}^{\infty} (-1)^k \frac{\sin k}{\sqrt{k^3}}$

16. We know that the Taylor series generated by e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Use the Ratio Test to show that this series converges for all x.

17. We know that the Taylor series generated by $\cos x$ is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$. Use the Ratio Test to show that this series converges for all x.

18. The power series $\sum_{n=0}^{\infty} \frac{(x-3)^n}{5^n}$ is geometric. Show that the information given by the Ratio Test about convergence agrees with what we know about the convergence of a geometric series.

19. What exactly do we mean when we write $\sum_{k=1}^{\infty} a_k = 4$? Your answer can be brief, but must be precise and accurate. You will get full credit only if you use words correctly.

20. Suppose that a power series of the form $\sum_{k=0}^{\infty} c_k(x-2)^k$ has a radius of convergence of 7. What are the possibilities for the interval of convergence of the series?

21. Suppose the interval of convergence of the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is $(-5, 7]$.

    (a) What is a?

    (b) Does the series converge for $x = 6.5$? For $x = -6.5$?

22. The interval of convergence of the Maclaurin series for $\ln(1+u)$ is $u \in (-1, 1]$. On this interval the series converges to $\ln(1+u)$.

    (a) Using any method you like, find the Maclaurin series for $\ln(1+u)$.

    (b) By setting $u = x - 1$ in part (a), find a power series expansion for $\ln x$ centered at $x = 1$.

    (c) Find the Taylor series for $\ln x$ at $x = 1$ by taking derivatives. Make sure your answers to parts (b) and (c) agree. (They ought to because *if* a function has a power series expansion in $(x-1)$ then that expansion will be the Taylor series about $x = 1$.)

23. In this problem you'll compare the error analysis arrived at using the alternating series error estimate with that gotten using the Taylor Remainder.

    Let $f(x) = x^{1/3}$.

    (a) Find the third order Taylor Polynomial, $T_3(x)$, for f at $x = 27$.

    (b) Use $T_3(x)$ found in part (a) to approximate $28^{1/3}$.

    (c) Find an upper bound for the error in this approximation by using the alternating series error estimate.

    (d) Now find an upper bound for the error in this approximation by using the Taylor Remainder (i.e., Taylor's Inequality).

24. Multiplying Series:

    (a) Find the third degree Taylor polynomial (centered at 0) for $f(x) = \sin x \cdot \cos x$ by multiplying Taylor series for $\sin x$ and $\cos x$ and stopping when you know that all remaining terms are of degree 4 or higher. Check your answer by using the identity $\sin 2x = 2 \sin x \cos x$.

    (b) In your last homework set you found the series for $\sqrt{1+x}$. (It was §8.8 #1: the answer is in the back of the book.) Multiply this series by itself to find the second degree Taylor polynomial for $\sqrt{1+x} \cdot \sqrt{1+x}$. Without computing, what do you expect the coefficient of the x^3 term to be?

25. Amanda asked her friend Charlie for some help when studying for a math test on series. Charlie had quick and easy methods, but he sometimes said things that are incorrect. Your job in this problem is to pay very careful attention to Amanda and Charlie's conversation and correct Charlie wherever necessary.

    (a) Amanda asked Charlie how you can tell if the series,

    $$1 + (-1) + (-1)^2 + (-1)^3 + (-1)^4 + ...$$

    converges or diverges. Charlie answered, "That's easy. This is just a geometric series with $a = 1$ and $r = -1$. So you just plug into the formula,

    $$\frac{a}{1-r} = \frac{1}{1-(-1)} = \frac{1}{2}.$$

    So, it converges to one half." Do you agree with Charlie's statement? Explain why or why not.

    (b) Amanda told Charlie that she was having a lot of trouble with geometric series. She asked Charlie how you could find out whether an expression like:

    $$3 + 3 \cdot 7^2 + 3 \cdot 7^4 + 3 \cdot 7^6 + ... + 3 \cdot 7^{20}$$

converged or not, and if it did, what it converged to. Charlie answered, "Okay, these are two step problems. First, that's a geometric series with $a = 3$, $r = 7^2$. So, to get the total, you just plug into the formula:

$$\frac{a(1-r^{n+1})}{1-r} = \frac{3(1-49^{21})}{1-49}."$$

Charlie told Amanda that you can work that out on a calculator. (Is Charlie's closed form correct? Explain below.) "You can tell the series doesn't converge" Charlie continued," because the r is 49. That's greater than one, so the series doesn't converge." Is Charlie's statement about convergence accurate? Why or why not?

(c) The last question that Amanda asked was about a general infinite series like $\sum_{k=1}^{\infty} a_k$. She said that she remembered hearing something about looking at the value of a_k when k gets really, really big, and that can tell you something about whether the series $\sum_{k=1}^{\infty} a_k$ converges or not. Charlie responded,"Oh yeah, this makes convergence and divergence really easy. If you look at what the formula for a_k is, and if that formula gets closer and closer to zero then the series converges. Otherwise, the series diverges." Do you think Charlie is gave Amanda very solid advice? What advice would you have given her?

26. Extra Credit: Stewart section 8.9 #25:
Hint: In #25 we are looking at the situation for which d is much much smaller than D, so $\frac{d}{D}$ is small. We would like to use only the first few terms of a Taylor series, so we need to write a series in u where u is very small. That's why the book suggests writing a series in powers of $u = \frac{d}{D}$. Begin by manipulating the expression given until you have

$$\frac{q}{D^2} - \frac{q}{D^2}\left[\frac{1}{\left(1+\frac{d}{D}\right)^2}\right] = \frac{q}{D^2}\left[1 - \frac{1}{\left(1+\frac{d}{D}\right)^2}\right].$$

You'll expand the expression $\frac{1}{\left(1+\frac{d}{D}\right)^2}$. This looks like $\frac{1}{(1+u)^2}$ where you can think of $\frac{d}{D}$ as u. You'll only use the first two terms of the series to get the desired result.